APPROXIMATE ANALYTICAL SOLUTIONS OF A CLASS OF NON-LINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH CONFORMABLE DERIVATIVE

by

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In this paper, it is presented that an approximate solution of a class of non-linear differential equations with conformable derivative under boundary conditions by using sinc-Galerkin method that is not used to approximately solve the class of the considered equation in the literature. In the method, the solution function is expressed as a finite series in terms of composite translated sinc functions and the unknown coefficients. The problem is reduced into a non-linear matrix-vector system via sinc grid points, and when this system is solved by using Newton's method, the unknown coefficients of the solution function are easily obtained. Also, error analysis and some test problems are presented to illustrate the applicability and accuracy of the proposed method.

Key words: *non-linearity, sinc-Galerkin method, conformable derivative fractional boundary value problems*

Introduction

Fractional calculus that might be considered as an extension of ordinary calculus has applications in many engineering and scientific disciplines. While the order of derivatives in the ordinary calculus is positive integer number, ones in the fractional calculus can be any real number. Fractional calculus is a useful tool in the modelling of many scientific phenomena such as earthquake engineering, biomedical engineering, image processing, and physics. In [1, 2], the authors studied in detail on fractional calculus and their applications.

A fractional derivative has some different definitions in the literature. Some of them are the Caputo derivative, the Riemann-Liouville derivative, Atangana-Baleanu-Caputo derivative, the Caputo-Fabrizio derivative, beta derivative and conformable derivative. Several applications for those derivatives are developed in references [3-13]. In this paper, conformable definition of fractional derivative defined in [14] is considered.

The conformable derivative has some beneficial properties when it is compared to other ones. For example, the conformable derivative of a constant function is zero while this feature is not available in many other definitions. Also, the known formula of the derivative of the product and the the quotient of two functions and the well-known chain rule available in the conformable derivative. The availability of these properties in this definition provides con-

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venience for calculating analytical or approximate solution of fractional differential equations with conformable derivative.

Fractional non-linear differential equations are widely used in many areas of engineering and scientific research. Therefore, it is very important to obtain solutions of these equations. But, generally, it is difficult or not possible to find analytical solutions of those. For this reason, to approximately solve ones, many numerical and approximate methods are developed in the literature. In this study, we use the sinc-Galerkin method that has almost not been developed for the class of fractional non-linear differential equations:

$$L\left[y(x)\right]=y''(x)+p(x)y'(x)+q(x)y(x)+ \\ +n_1(x)y^s y^{(\alpha)}(x)+n_2(x)y^r(x)=f(x),\ \ 0<\alpha\le 1 \tag{1}$$

subject to homogeneous boundary conditions

$$y(a)=0,\ \ y(b)=0 \tag{2}$$

If the boundary conditions are inhomogeneous in eq. (2), they are easily converted to homogeneous ones by using technique given by [15].

The main advantage of the proposed method is to give more efficient results than classical polynomial methods in the problems which include singularities at some points. Therefore, many studies in the literature used the proposed method [15-18].

Preliminaries and notation

In this section, some preliminaries and notations related to sinc basic functions and conformable derivative are given.

Definition 1. Let $\alpha \in (n, n+1]$, and f be an n – differentiable function at t, where $t>0$. Then the conformable fractional derivative of f of order α is defined:

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon} \tag{3}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

Remark 1. As a consequence of *Definition 1*, one can easily show:

$$T_\alpha(f)(t)=t^{(\alpha-\alpha)}f^{\alpha}(t)$$

where $\alpha \in (n, n+1]$ and f is $(n+1)$ differentiable at $t>0$.

Theorem 1. Let $\alpha \in (n, n+1]$ and f; g be and be α – differentiable at a point $t>0$. Then:

$$T_\alpha\left(t^p\right)=pt^{p-\alpha},\ \text{ for all }\ p\in\mathbb{R}$$

$$T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g),\ \ \textit{f}\text{or all }\ b\in\mathbb{R}$$

$$T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$$

$$T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)+fT_\alpha(g)}{g^2}$$

$$T_\alpha(\lambda)=0,\ \text{ for all constant functions }\ f(t)=\lambda$$

Definition 2. The sinc function is defined on the whole real line $-\infty < x < \infty$ by:

$$sinc(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Definition 3. For $h > 0$ and $k = 0, \pm 1, \pm 2,...$ the translated sinc functions with space node are given:

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right) = \begin{cases} \dfrac{\sin\left(\pi \dfrac{x-kh}{h}\right)}{\pi \dfrac{x-kh}{h}} & x \neq kh \\ 1 & x = kh \end{cases}$$

Definition 4. If $f(x)$ is defined on the real line, then for $h > 0$ the series:

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\, sinc\left(\frac{x-kh}{h}\right)$$

is called the Whittaker cardinal expansion of f whenever this series converges.

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct an approximation on the interval (a, b), the conformal map:

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right)$$

is employed. The basis functions on the interval (a, b) are derived from the composite translated sinc functions:

$$S_k(z) = S(k,h)(z) \circ \phi(z) = sinc\left[\frac{\phi(z)-kh}{h}\right]$$

The inverse map of $w = \phi(z)$ is:

$$z = \phi^{-1}(w) = \frac{a+be^{w}}{1+e^{w}}$$

The sinc grid points $z_k \in (a, b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted:

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \quad k = 0, \pm 1, \pm 2, ...$$

Theorem 2. Let Γ be $(0, 1)$, $F \in B(D_E)$, then for $h > 0$ sufficiently small:

$$\int_{\Gamma} F(z)\mathrm{d}z - h\sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D} \frac{F(z)k(\phi,h)(z)}{\sin\dfrac{\pi\phi(z)}{h}}\mathrm{d}z \equiv I_F \tag{4}$$

where

$$\left|k(\phi,h)\right|_{z\in\partial D} = \left|\mathrm{e}^{\left[\frac{i\pi\phi(z)}{h} sgn\left[Im\phi(z)\right]\right]}\right|_{z\in\partial D} = \mathrm{e}^{\frac{-\pi d}{h}}$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 3. If there exist positive constants α, β, and C such that:

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C\begin{cases} \mathrm{e}^{-\alpha|\phi(x)|} & x\in\psi\left[(-\infty,\infty)\right] \\ \mathrm{e}^{-\beta|\phi(x)|} & x\in\psi\left[(0,\infty)\right] \end{cases} \tag{5}$$

then the error bound for the quadrature rule eq. (4)

$$\left|\int_\Gamma F(x)dx - h\sum_{j=-M}^{N}\frac{(x_j)}{(x_j)}\right| \le C\left(\frac{\mathrm{e}^{-\alpha Mh}}{\alpha}+\frac{\mathrm{e}^{-\beta Nh}}{\beta}\right)+\left|I_F\right| \tag{6}$$

The infinite sum in eq. (4) is truncated with the use of eq. (5) to arrive at the inequality eq. (6). Making the selections:

$$h=\sqrt{\frac{\pi d}{\alpha M}} \text{ and } N\equiv\left[\left\lfloor\frac{\alpha M}{\beta}+1\right\rfloor\right]$$

where $[\lfloor\,.\,\rfloor]$ is the integer part of the statement and M is the integer value which specifies the grid size:

$$\int_\Gamma F(x)\mathrm{d}x = h\sum_{j=-M}^{N}\frac{F(x_j)}{(x_j)}+O\left(\mathrm{e}^{-(\pi\alpha dM)^{1/2}}\right) \tag{7}$$

These theorems are used for the integrals in the inner products that arise from the method presented here.

The solution method

An approximate solution of $y(x)$ in eq. (1) is represented:

$$y_n(x)=\sum_{k=-M}^{N}c_kS_k(x),\ \ n=M+N+1 \tag{8}$$

where S_k is function $S(k, h)\circ\phi(x)$ for some fixed step size h. The unknown coefficients c_k in eq. (8) are determined by orthogonalizing the residual with respect to the basis functions, for $k=-M,\dots, N$, *i. e.*

$$\langle y'',S_k\rangle+\langle p(x)y',S_k\rangle+\langle q(x)y,S_k\rangle+\langle n_1(x)y^s y^{(\alpha)},S_k\rangle+\langle n_2(x)y^r,S_k\rangle=\langle f(x),S_k\rangle \tag{9}$$

The inner product used for the sinc-Galerkin method is defined:

$$\langle f,g\rangle=\int_a^b f(x)g(x)w(x)\mathrm{d}x$$

where $w(x)$ a weight function which is taken for second-order boundary value problems in the form:

$$w(x)=\frac{1}{\phi'(x)}$$

We need the following theorems for the approximation of inner products in eq. (9).

Theorem 4. The following relations hold:

$$\left\langle y'', S_k \right\rangle \approx h \sum_{j=-M}^{N} \sum_{i=0}^{2} \frac{y(x_j)}{\phi'(x_j)h^i} \delta_{kj}^{(i)} g_{2,i}(x_j) \tag{10}$$

$$\left\langle p(x) y', S_k \right\rangle \approx -h \sum_{j=-M}^{N} \sum_{i=0}^{1} \frac{y(x_j)}{\phi'(x_j)h^i} \delta_{kj}^{(i)} g_{1,i}(x_j) \tag{11}$$

and for $G(x) = n_2(x)y^r(x)$, $G(x) = q(x)y(x)$, and $G(x) = f(x)$:

$$\langle G, S_k \rangle \approx h \frac{G(x_k) w(x_k)}{\phi'(x_k)} \tag{12}$$

The *Proof* of this theorem and values of $g_{k,i}(x)$ can be found in [17].

Theorem 5. The following relation holds:

$$\left\langle n_1(x) y^s y^{(\alpha)}, S_k \right\rangle \approx -\frac{h}{s+1} \cdot \sum_{j=-M}^{N} \frac{y^{s+1}(x_j)}{\phi'(x_j)} \left[\frac{1}{h} \delta_{kj}^{(1)} \left(\phi' n_1 w x^{1-\alpha} \right)(x_j) + \delta_{kj}^{(0)} \left(n_1 w x^{1-\alpha} \right)'(x_j) \right] \tag{13}$$

Proof. For $n_1(x)y^s y^{(\alpha)}$, the inner product with sinc basis elements is given:

$$\left\langle n_1 y^s y^{(\alpha)}, S_k \right\rangle = \int_a^b y^s x^{1-\alpha} y' (S_k n_1 w) \mathrm{d}x$$

Integrating by parts to remove the first derivative from the dependent variable y leads to the equality:

$$\left\langle n_1 y^s y^{1-\alpha}, S_k \right\rangle = B_1 - \frac{1}{s+1} \int_a^b y^{s+1} \left(S_k n_1 w x^{1-\alpha} \right)' \mathrm{d}x \tag{14}$$

where the boundary term

$$B_1 = \left[\frac{1}{s+1} \left(y^{s+1} S_k n_1 w x^{1-\alpha} \right) \right]_{x=a}^{b} = 0$$

and expanding the derivatives under the integral in eq. (14) yields:

$$\left\langle n_1 y^s y^{(\alpha)}, S_k \right\rangle = -\frac{1}{s+1} \int_a^b y^{s+1} \left[S_k^{(1)} \phi' \left(n_1 w x^{1-\alpha} \right) + S_k^{(0)} \left(n_1 w x^{1-\alpha} \right)' \right] \mathrm{d}x \tag{15}$$

Applying the sinc quadrature rule given by eq. (7) to the right-hand side of eq. (15) and deleting the error term yields eq. (13).

Replacing each term of (9) with the approximations defined in eqs. (10)-(13), respectively, and replacing $y(x_j)$ by c_j, and dividing by h, we obtain the theorem:

Theorem 6. If the assumed approximate solution of the boundary-value problem (1) is (8), then the discrete sinc-Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{N}$ is given, for $k = -M, \dots, N$:

$$\sum_{j=-M}^{N}\left\{\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}c_j-\sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\phi'(x_j)}c_j-\frac{1}{s+1}\left[\frac{1}{h^i}\delta_{kj}^{(1)}(n_1wx^{1-\alpha})(x_j)c_j^{s+1}+\right.\right.$$
$$\left.\left.+\frac{(n_1wx^{1-\alpha})(x_k)}{\phi'(x_k)}c_k^{s+1}\right]\right\}+\frac{\mu_0(x_k)w(x_k)}{\phi'(x_k)}c_k+\frac{n_2(x_k)w(x_k)}{\phi'(x_k)}c_k^r=\frac{f(x_k)w(x_k)}{\phi'(x_k)} \tag{16}$$

Now we define some notation represent the system (16) in matrix-vector form. Let $\mathbf{D}(y)$ is the diagonal matrix whose diagonal elements are $y(x_{-M})$, $y(x_{-M+1})$,..., $y(x_N)$ and non-diagonal elements are zero; also for $0 \le i \le 2$, let $\mathbf{I}^{(i)}$ denote the matrices:

$$\mathbf{I}^{(i)}=\left[\delta_{jk}^{(i)}\right],\quad j,k=-M,\ldots,N$$

where $\mathbf{I}$ and $\mathbf{D}$ are square matrices of dimension $n \times n$. In order to calculate the unknown coefficients c_k in the non-linear system (16), we rewrite this system using the aforementioned notations in matrix-vector form:

$$\mathbf{AC}+\mathbf{BC}^s+\mathbf{EC}^r=\mathbf{F} \tag{17}$$

where

$$\mathbf{A}=\sum_{j=0}^{2}\frac{1}{h^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{g_{2,j}}{\phi'}\right)-\sum_{j=0}^{1}\frac{1}{h^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{g_{1,j}}{\phi'}\right)+\mathbf{I}^{(0)}\mathbf{D}\left(\frac{g_{0,0}}{\phi'}\right)$$

$$\mathbf{B}=-\frac{1}{s+1}\left\{\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}\left(n_1wx^{1-\alpha}\right)+\mathbf{I}^{(0)}\mathbf{D}\left[\frac{\left(n_1wx^{1-\alpha}\right)'}{\phi'}\right]\right\}$$

$$\mathbf{E}=\mathbf{D}\left(\frac{n_2w}{\phi'}\right),\quad \mathbf{F}=\mathbf{D}\left(\frac{wf}{\phi'}\right),\quad \mathbf{C}^j=\left(c_{-M}^j,c_{-M+1}^j,\ldots,c_{N-1}^j,c_N^j\right)^T,\quad j=1,s,r$$

Now we have a non-linear system of n equations in n unknown coefficients given by eq. (17). Solving by Newton's method, we can obtain the unknown coefficients c_k that are necessary for approximating the solution in eq. (8).

Error estimation

In this section, we define two error functions that are actual error function $E_N(x)$ and estimate error function $\tilde{E}_N(x)$ to check the accuracy of the presented method. Actual error function $E_N(x)$ is used in the problems that are known the exact solutions and it is defined:

$$E_N(x)=\left|y_N(x)-y(x)\right| \tag{18}$$

where $y(x)$ is the exact solution of eq. (1). The estimate error function $\tilde{E}_N(x)$ might be used in the problems that are unknown the exact solutions. If the $y_N(x)$ is an approximate solution eq. (1), then when these functions and their derivatives are substituted into eq. (1) the obtained equation should be satisfied approximately. In short, for $x_k \in [a, b]$, the function $\tilde{E}_N(x)$ is defined:

$$\tilde{E}_N(x_k)=\left|L\left[y_N(x_k)\right]-f(x_k)\right|\cong 0 \tag{19}$$

and $\tilde{E}_N \leq 10^{-tk}$ (tk any positive contant). If max $10^{-tk} = 10^{-t}$ is prescribed, the truncation limit N is increased until the difference $\tilde{E}_N(x_k)$ at each of the points becomes smaller than the prescribed 10^{-t} [19].

Computational examples

In this section, two numerical examples are presented to show the accuracy of the present method. In the first example, a problem that has the known exact solution is selected. On this problem, the present method is tested via the error functions given by eqs. (18) and (19). In the second example, a singular problem that has the known exact solution for integer order derivative case is considered. So, the example is only tested the error function given by eq. (19). In the both examples, $N = M$ and $h = \pi/(2N)^{1/2}$ is taken and the obtained results are illustrated via tables and graphics at selected points in the interval (0, 1).

Example 1. Consider the following non-linear fractional boundary value problem:

$$y''(x)+x^2y^4(x)y^{(0.5)}(x)+xy^2(x)=f(x); \quad y(0)=y(1)=0 \tag{20}$$

where

$$f(x)=-2+6x+x^5-2x^6+x^7-2x^{11.5}+11x^{12.5}-24x^{13.5}+26x^{14.5}-14x^{15.5}+3x^{16.5}$$

The exact solution of eq. (20) $y(x) = x^2(x - 1)$ is given by. Table 1 presents numerical values of the actual error functions obtained for eq. (20) when N = 16, 32, 64, and 128. Similarly, in tab. 2, numerical values of the estimated error functions obtained for eq. (20) when N = 16, 32, 64, and 128 are presented. Numerical results of the exact solution and the approximate solutions obtained by the present method of eq. (20) for N = 4, 16, and 64 are given in tab. 3. In fig. 1, it is given that graphics of exact solution and approximate solution obtained by the present method in the interval (0, 1) for N = 16, 32, and 64.

Table 1. Numerical results of the error function E_N for *Example 1* for different values of N

x	E_{16}	E_{32}	E_{64}	E_{128}
0.2	$1.629 \cdot 10^{-5}$	$4.855 \cdot 10^{-7}$	$2.941 \cdot 10^{-9}$	$2.006 \cdot 10^{-12}$
0.4	$3300 \cdot 10^{-5}$	$9.689 \cdot 10^{-7}$	$5.888 \cdot 10^{-9}$	$4.015 \cdot 10^{-12}$
0.6	$5.025 \cdot 10^{-5}$	$1.456 \cdot 10^{-7}$	$8.846 \cdot 10^{-9}$	$6.038 \cdot 10^{-12}$
0.8	$6.754 \cdot 10^{-5}$	$1.953 \cdot 10^{-7}$	$1.187 \cdot 10^{-9}$	$8.104 \cdot 10^{-12}$

Table 2. Numerical results of the error function $\tilde{E}_N$ for *Example 1* for different values of N

x	$\tilde{E}_{16}$	$\tilde{E}_{32}$	$\tilde{E}_{64}$	$\tilde{E}_{128}$
0.2	$3.659 \cdot 10^{-4}$	$3.306 \cdot 10^{-6}$	$2.515 \cdot 10^{-9}$	$1.062 \cdot 10^{-12}$
0.4	$1.605 \cdot 10^{-4}$	$3.306 \cdot 10^{-6}$	$5.958 \cdot 10^{-9}$	$3.572 \cdot 10^{-12}$
0.6	$1.058 \cdot 10^{-4}$	$3.360 \cdot 10^{-6}$	$6.491 \cdot 10^{-9}$	$1.063 \cdot 10^{-12}$
0.8	$3.993 \cdot 10^{-4}$	$3.815 \cdot 10^{-6}$	$3.860 \cdot 10^{-9}$	$7.872 \cdot 10^{-12}$

Table 3. Numerical results of approximate and exact solutions for *Example 1* for different values of N

x	$N = 4$	$N = 16$	$N = 64$	Exact
0.2	–0.0310262	–0.0319837	–0.0319999	–0.0320000
0.4	–0.0947633	– 0.0959669	–0.0959999	–0.0960000
0.6	–0.1405928	– 0.1439497	–0.1439999	– 0.1440000
0.8	–0.1247026	– 0.1279324	–0.1279999	–0.1280000

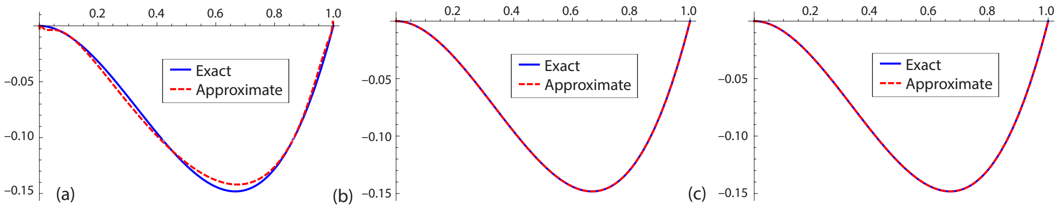

Figure 1. Graphics of exact and approximate solutions for *Example 1* when N = 4, 16, and 64; (a) N = 4, (b) N = 16, and (c) N = 64 (*for color image see journal web site*)

Example 2. Consider the non-linear singular boundary value problem:

$$y''(x)-\frac{1}{x}y'(x)+\frac{1}{x(x-1)}y^3(x)y^{(\alpha)}(x)-\frac{1}{x-1}y^2(x)=f(x);\quad y(0)=y(1)=0 \tag{21}$$

where

$$f(x)=\pi\cos\pi x-\frac{\sin\pi x}{x}-\pi^2 x\sin\pi x-\frac{x^2(\sin\pi x)^2}{x-1}+\frac{\pi x^3\cos\pi x(\sin\pi x)^3}{x-1}+\frac{x^2(\sin\pi x)^4}{x-1}$$

The exact solution of eq. (21) for α = 1 is $y(x)$ = sinπx. Table 4 presents numerical results obtained for α = 0.1, 0.5, 0.9, 1 in eq. (21) when N = 128. In tab. 5, numerical results of the estimated error functions obtained for α = 0.5 when N = 16, 32, 64, and 128 are presented. Similarly, in tab. 6, numerical results of the estimated error functions obtained for α = 0.2, 0.4, 0.6, 0.8 when N = 128 are given. In fig. 2(a), it is given that graphics of approximate solutions obtained by the present method for α = 0.1, 0.5, 0.9, 1 in the interval (0.4, 0.41) when N = 128. In the interval (0.2, 0.8), fig. 2(b) presents graphics of the estimated error functions for N = 4, 16, and 64 when α = 0.7 in eq. (21).

Table 4. Numerical results for different values of α when N = 128 for *Example 2*

x	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 1$
0.2	0.119456	0.118795	0.117849	0.117557
0.4	0.386902	0.384623	0.381404	0.380423
0.6	0.577281	0.575024	0.571677	0.570634
0.8	0.472163	0.471740	0.470637	0.470228

Table 5. Numerical results of $\tilde{E}_N$ for different values of N when α = 0.5 in *Example 2*

x	$\tilde{E}_{16}$	$\tilde{E}_{32}$	$\tilde{E}_{64}$	$\tilde{E}_{128}$
0.2	$2.219 \cdot 10^{-2}$	$4.308 \cdot 10^{-4}$	$2.956 \cdot 10^{-6}$	$4.998 \cdot 10^{-10}$
0.4	$1.519 \cdot 10^{-2}$	$2.503 \cdot 10^{-2}$	$3.943 \cdot 10^{-2}$	$1.842 \cdot 10^{-10}$
0.6	$1.088 \cdot 10^{-2}$	$1.715 \cdot 10^{-2}$	$1.037 \cdot 10^{-2}$	$1.595 \cdot 10^{-10}$
0.8	$2.613 \cdot 10^{-2}$	$7.256 \cdot 10^{-2}$	$3.504 \cdot 10^{-2}$	$7.912 \cdot 10^{-10}$

Table 6. Numerical results of $\tilde{E}_{128}$ for different values of α in *Example 2*

x	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
0.2	$8.567 \cdot 10^{-10}$	$6.108 \cdot 10^{-10}$	$4020 \cdot 10^{-10}$	$2.554 \cdot 10^{-10}$
0.4	$3.707 \cdot 10^{-10}$	$2.374 \cdot 10^{-10}$	$1.411 \cdot 10^{-10}$	$8.948 \cdot 10^{-10}$
0.6	$3.137 \cdot 10^{-10}$	$2.118 \cdot 10^{-10}$	$1.256 \cdot 10^{-10}$	$1.340 \cdot 10^{-10}$
0.8	$2.864 \cdot 10^{-10}$	$1.754 \cdot 10^{-10}$	$8.116 \cdot 10^{-10}$	$1.491 \cdot 10^{-10}$

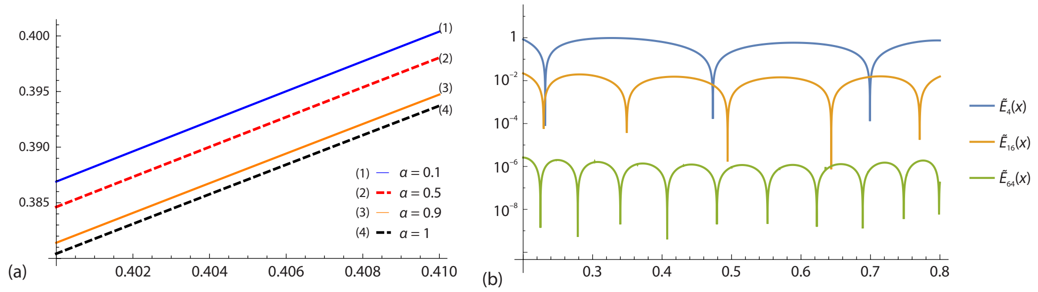

Figure 2. Graphics of the estimated error functions and approximate solutions for *Example 2*; (a) zoomed graphics of approximate solutions for different values of α in *Example 2* when $N = 128$ and (b) graphics of the estimated error functions $\widetilde{E}_N$ for $\alpha = 0.7$ in *Example 2* when $N = 4, 16, 64$ (*for color image see journal web site*)

Conclusion

In this paper, the sinc-Galerkin method is proposed for the numerical solutions of a class of non-linear differential equations. In order to illustrate the accuracy of the presented method, the approximate results are compared with exact results. Those comparisons imply that the sinc-Galerkin method provides a good approximate solution and has a capability for solving other types of non-linear differential equations which occur in thermal physics.

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