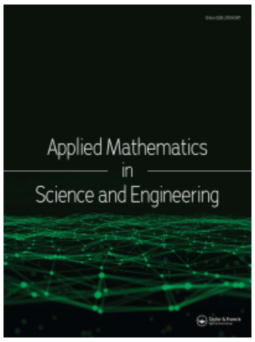

Applied Mathematics in Science and Engineering

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gipe21

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To cite this article: Waseem Ahmad Khan, Jihad Younis, Ugur Duran & Azhar Iqbal (2022) The higher-order type 2 Daehee polynomials associated with p-adic integral on $\mathbb{Z}_p$, Applied Mathematics in Science and Engineering, 30:1, 573-582, DOI: 10.1080/27690911.2022.2114470

To link to this article: https://doi.org/10.1080/27690911.2022.2114470

Published online: 24 Aug 2022.

APPLIED MATHEMATICS IN SCIENCE AND ENGINEERING
2022, VOL. 30, NO. 1, 573–582
https://doi.org/10.1080/27690911.2022.2114470

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The higher-order type 2 Daehee polynomials associated with p-adic integral on $\mathbb{Z}_p$

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ABSTRACT
In this paper, the higher-order type 2 Daehee polynomials are introduced and some of their relations and properties are derived. Then, some p-adic integral representations of not only higher-order type 2 Daehee polynomials and numbers but also type 2 Daehee polynomials and numbers are acquired. Several identities and relations related to both central factorial numbers of the second kind and Stirling numbers of the first and second kinds are investigated. Moreover, the conjugate higher-order type 2 Daehee polynomials are considered and some correlations covering the type 2 Daehee polynomials of order β and the conjugate higher-order type 2 Daehee polynomials are attained.

ARTICLE HISTORY
Received 30 May 2022
Accepted 13 August 2022

KEYWORDS
Type 2 Daehee polynomials; higher-order type 2 Daehee polynomials; p-adic integral; the central factorial numbers of the second kind; Stirling numbers of the first and second kinds

1991 MATHEMATICS SUBJECT CLASSIFICATIONS
11B73; 11B83; 05A19; 11B68; 33C45

1. Introduction

Recently, Kim et al. [1] considered the higher-order type 2 Bernoulli polynomials of the second kind as follows

$$\sum_{n=0}^{\infty} b_n^{*(r)}(\gamma)\frac{z^n}{n!} = \left(\frac{(1+z)-(1+z)^{-1}}{2\log(1+z)}\right)^r (1+z)^{\gamma} \tag{1}$$

and investigated several relations and formulae associated with central factorial numbers of the second kind and the higher-order type 2 Bernoulli polynomials. Inspired and motivated by the above study, here we consider the higher-order type 2 Daehee polynomials and derive some of their relations and properties. Also, we provide p-adic integral representations of type 2 Daehee polynomials and their higher-order polynomials. We then investigate some identities and relations. Moreover, we consider the conjugate type 2 Daehee polynomials of order β and acquire relationships including the type 2 Daehee polynomials of order β and the conjugate higher-order type 2 Daehee polynomials.

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Let $\mathbb{Z}_p = \{\gamma \in \mathbb{Q}_p : |\gamma|_p \le 1\}$ in conjunction with $\mathbb{Q}_p = \{\gamma = \sum_{n=-k}^{\infty} a_n p^n : 0 \le a_i \le p-1\}$ and $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$, cf. [2–10], where p be a prime number and the normalized p-adic absolute value is provided by $|p|_p = \frac{1}{p}$. For $g : \mathbb{Z}_p \to \mathbb{C}_p$ (g being a continuous map), the p-adic bosonic integral of g is given as follows:

$$I_0(g) := \int_{\mathbb{Z}_p} g(\gamma)\,\mathrm{d}\mu_0(\gamma) = \lim_{m\to\infty} \frac{1}{p^m} \sum_{\gamma=0}^{p^m-1} g(\gamma). \tag{2}$$

It is observed from (2) that

$$I_0(g_1) - I_0(g) = g^{'}(0), \tag{3}$$

where $g_1(\gamma) = g(\gamma+1)$ and $g'(l) = \frac{\mathrm{d}g(\gamma)}{\mathrm{d}\gamma}|_{\gamma=l}$, cf. [2–10].

The familiar Bernoulli polynomials are defined as follows (cf. [1,6,7,11–18])

$$\sum_{n=0}^{\infty} B_n(\gamma)\frac{z^n}{n!} = \frac{z}{\mathrm{e}^z - 1}\mathrm{e}^{\gamma z} = \int_{\mathbb{Z}_p} \mathrm{e}^{(\gamma+y)}z\mathrm{d}\mu_0(y).$$

The type 2 Bernoulli polynomials $b_n(\gamma)$ are given as follows (cf. [1,14,19])

$$\sum_{n=0}^{\infty} b_n(\gamma)\frac{z^n}{n!} = \frac{z}{\mathrm{e}^z - \mathrm{e}^{-z}}\mathrm{e}^{\gamma z}. \tag{4}$$

When $\gamma = 0$, we acquire $b_n(0) := b_n$ termed the type 2 Bernoulli numbers. We note $b_n(\gamma) = 2^{n-1}B_n(\frac{\gamma+1}{2})$ for $n \ge 0$.

The cosecant polynomials are defined by

$$\sum_{n=0}^{\infty} \mathcal{D}_n(\gamma)\frac{z^n}{n!} = \frac{z\,\mathrm{e}^{\gamma z}}{\sinh z} = \frac{2z\,\mathrm{e}^{\gamma z}}{\mathrm{e}^z - \mathrm{e}^{-z}}. \tag{5}$$

In this particular case $\gamma = 0$, $\mathcal{D}_n(0) := \mathcal{D}_n$ are termed the cosecant numbers that are a hot topic and have been worked in [1,14,19]. Here we observe that $\mathcal{D}_n(\gamma) = 2b_n(\gamma) = 2^n B_n(\frac{\gamma+1}{2})$ for $n \ge 0$. The sums of powers of consecutive integers can be computed by the Bernoulli polynomials as follow:

$$\sum_{l=0}^{n-1} l^r = \frac{B_{r+1}(n) - B_{r+1}(0)}{r+1} \quad (n \in \mathbb{N},\ r \in \mathbb{N}_0) \tag{6}$$

and it is noted that (cf. [1,14,19])

$$\sum_{l=0}^{n-1} (2l+1)^r = \frac{1}{2(r+1)}(\mathcal{D}_{r+1}(2n) - \mathcal{D}_{r+1}). \tag{7}$$

The higher-order type 2 Bernoulli polynomials are defined as follows:

$$\sum_{n=0}^{\infty} b_n^{(r)}(\gamma)\frac{z^n}{n!} = \left(\frac{z}{\mathrm{e}^z - \mathrm{e}^{-z}}\right)^r \mathrm{e}^{\gamma z}. \tag{8}$$

The Stirling numbers $S_2(n,r)$ of the second kind are given by (cf. [9,13,20–22])

$$\sum_{n=r}^{\infty} S_2(n,r)\frac{z^n}{n!} = \frac{(e^z-1)^r}{r!} \quad (r \geq 0) \tag{9}$$

and the Stirling numbers $S_1(n,r)$ of the first kind are provided by (cf. [2,13,19])

$$\sum_{n=r}^{\infty} S_1(n,r)\frac{z^n}{n!} = \frac{(\log(1+z))^r}{r!}, \tag{10}$$

which satisfies

$$(\gamma)_n = \sum_{r=0}^{n} S_1(n,r)\gamma^r. \tag{11}$$

The central factorial numbers $T(n,r)$ of the second kind are defined by (cf. [17,23–25])

$$\theta^n = \sum_{r=0}^{n} T(n,r)\theta^{[r]} \quad (n,r \geq 0), \tag{12}$$

where $\theta^{[r]} := \theta(\theta+\frac{r}{2}-1)(\theta+\frac{r}{2}-2)\cdots(\theta+\frac{r}{2}-(r-1))$ for $r \geq 1$ and $\theta^{[0]} := 1$. By (12), the generating function of $T(n,r)$ is provided by (cf. [23])

$$\sum_{n=r}^{\infty} T(n,r)\frac{z^n}{n!} = \frac{\left(e^{\frac{z}{2}}-e^{-\frac{z}{2}}\right)^r}{r!} \quad (r \geq 0). \tag{13}$$

Note that $T(n,r) = 0$ for $n < r$.

2. Higher-Order type 2 Daehee polynomials

The familiar Daehee polynomials $D_n(\gamma)$ are introduced by (cf. [2,3,5,8–10,21,26]):

$$\sum_{n=0}^{\infty} D_n(\gamma)\frac{z^n}{n!} = \frac{\log(1+z)}{z}(1+z)^{\gamma}. \tag{14}$$

In this particular case $\gamma = 0$, $D_n(0) := D_n$ are termed the Daehee numbers. By the formula (2) and (14), we have

$$\sum_{n=0}^{\infty} D_n(\gamma)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+z)^{\gamma+y}\, d\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (\gamma+y)_n\, d\mu_0(y)\frac{z^n}{n!}, \tag{15}$$

where $(\alpha)_n := \alpha(\alpha-1)\cdots(\alpha-n+1)$ for $n \geq 1$ with $(\alpha)_0 = 1$.

By (15), it is readily seen that

$$D_n(\gamma) = \int_{\mathbb{Z}_p} (\gamma+y)_n\, d\mu_0(y) \quad (n \geq 0).$$

The usual higher-order Daehee polynomials are introduced by (cf. [8,27])

$$\sum_{n=0}^{\infty} D_n^{(r)}(\gamma)\frac{z^n}{n!} = (1+z)^{\gamma}\frac{(\log(1+z))^r}{z^r}. \tag{16}$$

The following relation holds (cf. [8,27])

$$D_n^{(r)}(\gamma) = \sum_{m=0}^{n} B_m^{(r)}(\gamma) S_1(n,m).$$

The exponential generating functions of type 2 Daehee polynomials $d_n(\gamma)$ and numbers d_n are given by (cf. [9])

$$\frac{(1+z)^{\gamma}\log(1+z)}{(1+z)-(1+z)^{-1}} = \sum_{n=0}^{\infty} d_n(\gamma)\frac{z^n}{n!} \tag{17}$$

and

$$\frac{\log(1+z)}{(1+z)-(1+z)^{-1}} = \sum_{n=0}^{\infty} d_n \frac{z^n}{n!}. \tag{18}$$

We readily observe that $d_n(0) = d_n$. In [9], Kim et al. analyzed diverse relationships and properties of these polynomials and numbers by using their generating functions.

Now, we aim to investigate more properties and representations of the mentioned numbers and polynomials. We first compute, from (3) and (18), the following bosonic p-adic integrals

$$\int_{\mathbb{Z}_p} (1+z)^{2y+1+\gamma}\, \mathrm{d}\mu_0(y) = \frac{2\log(1+z)(1+z)^{\gamma}}{(1+z)-(1+z)^{-1}}$$

and

$$\int_{\mathbb{Z}_p} (1+z)^{2y+1+\gamma}\, \mathrm{d}\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (2y+1+\gamma)_n\, \mathrm{d}\mu_0(y)\frac{z^n}{n!},$$

which means

$$\sum_{n=0}^{\infty} \frac{1}{2}\int_{\mathbb{Z}_p} (2y+1+\gamma)_n\, \mathrm{d}\mu_0(y)\frac{z^n}{n!} = \sum_{n=0}^{\infty} d_n(\gamma)\frac{z^n}{n!}.$$

Thus, we acquire the Volkenborn integral representations of $d_n(\gamma)$ as given below.

Theorem 2.1: *The following Volkenborn integral representation of* $d_n(\gamma)$

$$d_n(\gamma) = \frac{1}{2}\int_{\mathbb{Z}_p} (2y+1+\gamma)_n\, \mathrm{d}\mu_0(y)$$

holds for $n \geq 0$ *and in addition, utilizing* (11), *the following relation*

$$d_n(\gamma) = \sum_{m=0}^{n} S_1(n,m) 2^m B_m\left(\frac{1+\gamma}{2}\right) \tag{19}$$

holds for $n \geq 0$.

Remark 2.1: The following p-adic integral representation

$$d_n = \frac{1}{2}\int_{\mathbb{Z}_p} (2y+1)_n \, \mathrm{d}\mu_0(y)$$

holds for $n \geq 0$.

Kim and Kim [9] introduced the type 2 Daehee polynomials of order $\beta \in \mathbb{R}$ denoting the set of all real numbers by

$$\sum_{n=0}^{\infty} d_n^{(\beta)}(\gamma)\frac{z^n}{n!} = \frac{(1+z)^{\gamma}\left(\log(1+z)\right)^{\beta}}{\left((1+z)-(1+z)^{-1}\right)^{\beta}}. \tag{20}$$

In this particular case $\gamma = 0$, $d_n^{(\beta)}(0) := d_n^{(\beta)}$ are termed the type 2 Daehee numbers of order β.

By means of (20) and choosing $\beta = r \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} d_n^{(r)}(\gamma)\frac{z^n}{n!} = (1+z)^{\gamma}\frac{\left(\log(1+z)\right)^{r}}{\left((1+z)-(1+z)^{-1}\right)^{r}}. \tag{21}$$

If we change z by $\mathrm{e}^{\frac{z}{2}} - 1$ in (21), we then acquire

$$\left(\frac{z}{\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}}}\right)^{r} \mathrm{e}^{\frac{\gamma z}{2}} = \sum_{m=0}^{\infty} d_m^{(r)}(\gamma)\frac{\left(\mathrm{e}^{\frac{z}{2}}-1\right)^{m}}{m!} = \sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{m=0}^{n} d_m^{(r)}(\gamma)S_2(n,m)\right)\frac{z^m}{m!} \tag{22}$$

and also

$$\left(\frac{z}{\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}}}\right)^{r} \mathrm{e}^{\frac{\gamma z}{2}} = \sum_{n=0}^{\infty} b_n^{(r)}\frac{z^n}{n!}\sum_{m=0}^{\infty}\gamma^m\frac{z^m}{2^m m!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{1}{2^m}b_{n-m}^{(r)}\gamma^m\right)\frac{z^n}{n!}. \tag{23}$$

Thus, by means of (22) and (23), we provide the following relation.

Theorem 2.2: *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{1}{2^m}b_{n-m}^{(r)}\gamma^m = \frac{1}{2^n}\sum_{m=0}^{n} d_m^{(r)}(\gamma)S_2(n,m).$$

For $r \in \mathbb{N}_0$, upon setting $\beta = -r$ and changing z by $\mathrm{e}^{\frac{z}{2}} - 1$ in (21), we then investigate

$$\mathrm{e}^{\frac{z}{2}\gamma}\left(\frac{\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}}}{z}\right)^{r} = \sum_{l=0}^{\infty} d_l^{(-r)}(\gamma)\frac{1}{l!}(\mathrm{e}^{\frac{z}{2}}-1)^l = \sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{l=0}^{n} d_l^{(-r)}(\gamma)S_2(n,l)\right)\frac{z^n}{n!}$$

and also

$$\frac{\left(\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}}\right)^{r}}{z^r}\mathrm{e}^{\frac{z}{2}\gamma} = \frac{r!}{z^r}\mathrm{e}^{\frac{z}{2}\gamma}\frac{(\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}})^r}{r!} = \frac{r!}{z^r}\sum_{n=0}^{\infty}\frac{\gamma^n z^n}{2^n}\sum_{l=r}^{\infty}\frac{z^l}{l!}T(l,r)$$

$$= r! \sum_{n=0}^{\infty} \frac{\gamma^n z^n}{2^n} \sum_{l=0}^{\infty} \frac{z^l T(l+r,r)}{(l+r)!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} T(l+r,r) \binom{n}{l} \frac{\gamma^{n-l} 2^{-n+l}}{\binom{l+r}{l}} \right) \frac{z^n}{n!}.$$

Thereby, we give the following result.

Theorem 2.3: *For $n, r \in \mathbb{N}_0$, we have*

$$\sum_{l=0}^{n} \binom{n}{l} \frac{2^l T(l+r,r) \gamma^{n-l}}{\binom{l+r}{l}} = \sum_{l=0}^{n} d_l^{(-r)}(\gamma) S_2(n,l)$$

and particularly,

$$T(n+r,r) = \frac{\binom{n+r}{n}}{2^n} \sum_{l=0}^{n} d_l^{(-r)} S_2(n,l) \quad \text{and} \quad d_l^{(-r)} = \sum_{l=0}^{n} \binom{n}{l} \frac{2^l S_1(n,l)}{\binom{l+r}{l}}.$$

If we change z by $2\log(1+z)$ in (13), we observe that

$$\left((1+z) - (1+z)^{-1}\right)^r \frac{1}{r!} = \sum_{l=r}^{\infty} \frac{(\log(1+z))^l}{l!} T(l,r) 2^l$$

$$= \sum_{l=r}^{\infty} T(l,r) \sum_{n=l}^{\infty} 2^l S_1(n,l) \frac{z^n}{n!} = \sum_{n=r}^{\infty} \left(\sum_{l=r}^{n} S_1(n,l) T(l,r) 2^l \right) \frac{z^n}{n!}$$

and

$$\left((1+z) - (1+z)^{-1}\right)^r \frac{1}{r!} = \frac{\left(\log(1+z)\right)^{-r} (\log(1+z))^r}{r! \left((1+z) - (1+z)^{-1}\right)^{-r}}$$

$$= \sum_{m=r}^{\infty} S_1(m,r) \frac{z^m}{m!} \sum_{l=0}^{\infty} d_l^{(-r)} \frac{z^l}{l!} = \sum_{n=r}^{\infty} \left(\sum_{m=r}^{n} \binom{n}{m} S_1(m,r) d_{n-m}^{(-r)} \right) \frac{z^n}{n!},$$

which provide the following relationship.

Theorem 2.4: *The following relationship*

$$\sum_{l=r}^{n} S_1(n,l) T(l,r) 2^l = \sum_{l=r}^{n} \binom{n}{l} S_1(l,r) d_{n-l}^{(-r)}$$

holds for $n, r \geq 0$.

Note that the higher-order cosecant polynomials are defined by (see [5,8,16])

$$\sum_{n=0}^{\infty} \mathcal{D}_n^{(\beta)}(\gamma) \frac{z^n}{n!} = \left(\frac{2z}{e^z - e^{-z}} \right)^{\beta} e^{\gamma z}. \tag{24}$$

In this special case $\gamma = 0, \mathcal{D}_n^{(\beta)}(0) := \mathcal{D}_n^{(\beta)}$ are termed the higher-order cosecant numbers.

If we change z by $\log(1+z)$ in (24), we then obtain

$$(1+z)^{\gamma} \frac{\left(\log(1+z)\right)^{\beta}}{\left((1+z)-(1+z)^{-1}\right)^{\beta}} = \sum_{m=0}^{\infty} 2^{\beta} \mathcal{D}_m^{(\beta)}(\gamma) \frac{(\log(1+z))^m}{m!}$$

$$= \sum_{m=0}^{\infty} 2^{\beta} \mathcal{D}_m^{(\beta)}(\gamma) \sum_{n=m}^{\infty} S_1(n,m) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(2^{\beta} \sum_{m=0}^{n} S_1(n,m) \mathcal{D}_m^{(\beta)}(\gamma) \right) \frac{z^n}{n!},$$

which means the following result.

Theorem 2.5: *The following correlation*

$$d_n^{(\beta)}(\gamma) = 2^{\beta} \sum_{m=0}^{n} \mathcal{D}_m^{(\beta)}(\gamma) S_1(n,m)$$

holds for $n \geq 0$ and $\beta \in \mathbb{R}$.

Kim-Kim [9] defined the higher-order type 2 Bernoulli polynomials by

$$\sum_{n=0}^{\infty} b_n^{(\beta)}(\gamma) \frac{z^n}{n!} = \left(\frac{z}{e^z - e^{-z}} \right)^{\beta} e^{\gamma z}. \tag{25}$$

In this particular case $\gamma = 0$, $b_n^{(\beta)}(0) := b_n^{(\beta)}$ are termed the higher-order type 2 Bernoulli numbers.

If we change z by $\log(1+z)$ in (25), we then attain

$$\frac{\left(\log(1+z)\right)^{\beta} (1+z)^{\gamma}}{\left((1+z)-(1+z)^{-1}\right)^{\beta}} = \sum_{m=0}^{\infty} \frac{(\log(1+z))^m}{m!} b_m^{(\beta)}(\gamma)$$

$$= \sum_{m=0}^{\infty} b_m^{(\beta)}(\gamma) \sum_{n=m}^{\infty} S_1(n,m) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_1(n,m) b_m^{(\beta)}(\gamma) \right) \frac{z^n}{n!}.$$

and also

$$\sum_{n=0}^{\infty} d_n^{(\beta)}(\gamma) \frac{z^n}{n!} = \frac{\left(\log(1+z)\right)^{\beta} (1+z)^{\gamma}}{\left((1+z)-(1+z)^{-1}\right)^{\beta}},$$

which means the following relationship.

Theorem 2.6: *The following relationship*

$$d_n^{(\beta)}(\gamma) = \sum_{m=0}^{n} b_m^{(\beta)}(\gamma) S_1(n,m)$$

is valid for $\beta \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

It is observed that

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}(1+z)^{(\gamma_1+\cdots+\gamma_r)+r+\gamma}\,\mathrm{d}\mu_0(\gamma_1)\,\mathrm{d}\mu_0(\gamma_2)\cdots\mathrm{d}\mu_0(\gamma_r)$$

$$=\frac{\big(\log(1+z)\big)^r}{\big((1+z)-(1+z)^{-1}\big)^r}(1+z)^\gamma=\sum_{n=0}^{\infty}d_n^{(r)}(\gamma)\frac{z^n}{n!},$$

which gives

$$\frac{d_n^{(r)}(\gamma)}{n!}=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}\binom{(\gamma_1+\cdots+\gamma_r)+r+\gamma}{n}\,\mathrm{d}\mu_0(\gamma_1)\cdots\mathrm{d}\mu_0(\gamma_r).$$

Here, we define the conjugate higher-order type 2 Daehee polynomials by

$$\sum_{n=0}^{\infty}\widehat{d}_n^{(\beta)}(\gamma)\frac{z^n}{n!}=\frac{(1+z)^\gamma\,\big((1+z)\log(1+z)\big)^\beta}{\big((1+z)-(1+z)^{-1}\big)^\beta}. \tag{26}$$

In this particular case $\gamma=0$, $\widehat{d}_n^{(r)}(0):=\widehat{d}_n^{(r)}$ are termed the conjugate higher-order type 2 Daehee numbers. By means of (26), we derive

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}(1+z)^{-(\gamma_1+\cdots+\gamma_r)+\gamma}\,\mathrm{d}\mu_0(\gamma_1)\cdots\mathrm{d}\mu_0(\gamma_r)$$

$$=(1+z)^\gamma\left(\frac{(1+z)\log(1+z)}{(1+z)-(1+z)^{-1}}\right)^r=\sum_{n=0}^{\infty}\widehat{d}_n^{(r)}(\gamma)\frac{z^n}{n!},$$

which means

$$\frac{1}{n!}\widehat{d}_n^{(r)}(\gamma)=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}\binom{-(\gamma_1+\cdots+\gamma_r)+\gamma}{n}\,\mathrm{d}\mu_0(\gamma_1)\cdots\mathrm{d}\mu_0(\gamma_r). \tag{27}$$

By formula (27), it is readily seen that

$$\frac{1}{n!}\widehat{d}_n^{(r)}(r)=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}\binom{-(\gamma_1+\cdots+\gamma_r)+\gamma}{n}\,\mathrm{d}\mu_0(\gamma_1)\cdots\mathrm{d}\mu_0(\gamma_r)$$

$$=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}\binom{(\gamma_1+\cdots+\gamma_r)+\gamma}{n}(-1)^n\,\mathrm{d}\mu_0(\gamma_1)\cdots\mathrm{d}\mu_0(\gamma_r)$$

$$=\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{(\gamma_1+\cdots+\gamma_r)+\gamma}{n}(-1)^n\,\mathrm{d}\mu_0(\gamma_1)\cdots\mathrm{d}\mu_0(\gamma_r)$$

$$= \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{(-1)^n}{m!} d_m^{(r)},$$

which implies the following formulas.

Theorem 2.7: *Each of the following relations*

$$\sum_{m=1}^{n} \binom{n-1}{n-m} \frac{(-1)^n}{m!} d_m^{(r)} = \frac{\widehat{d}_n^{(r)}(r)}{n!}$$

and

$$\sum_{m=1}^{n} \binom{n-1}{n-m} \frac{(-1)^n}{m!} \widehat{d}_n^{(r)} = \frac{d_m^{(r)}(r)}{n!}$$

is valid for $n, r \in \mathbb{N}_0$.

3. Conclusion

In this paper, the higher-order type 2 Daehee polynomials have been studied and several of their relations and properties have been derived. Some p-adic integral representations of type 2 Daehee polynomials and the higher-order type 2 Daehee polynomials have been acquired. Then, diverse identities and relations related to the central factorial numbers of the second and the Stirling numbers of the second and the first kinds have been investigated. Moreover, the conjugate higher-order type 2 Daehee polynomials have been considered and two relationships including the type 2 Daehee polynomials of order β and the conjugate higher-order type 2 Daehee polynomials have been provided.

Disclosure statement

No potential conflict of interest was reported by the authors.

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