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On (p, q)-Sine and (p, q)-Cosine Fubini Polynomials

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Abstract: In recent years, (p, q)-special polynomials, such as (p, q)-Euler, (p, q)-Genocchi, (p, q)-Bernoulli, and (p, q)-Frobenius-Euler, have been studied and investigated by many mathematicians, as well physicists. It is important that any polynomial have explicit formulas, symmetric identities, summation formulas, and relations with other polynomials. In this work, the (p, q)-sine and (p, q)-cosine Fubini polynomials are introduced and multifarious abovementioned properties for these polynomials are derived by utilizing some series manipulation methods. (p, q)-derivative operator rules and (p, q)-integral representations for the (p, q)-sine and (p, q)-cosine Fubini polynomials are also given. Moreover, several correlations related to both the (p, q)-Bernoulli, Euler, and Genocchi polynomials and the (p, q)-Stirling numbers of the second kind are developed.

Keywords: (p, q)-numbers; (p, q)-sine polynomials; (p, q)-cosine polynomials; (p, q)-special polynomials; (p, q)-Fubini polynomials; (p, q) Stirling numbers of the second kind

MSC: 33D15; 11B68; 11B73; 11B75; 11B83; 05A30

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1. Introduction

In recent years, (p, q)-calculus has been studied and examined widely by many physicists and mathematicians [1–12] (see also the references cited therein). (p, q)-special polynomials, such as (p, q)-Euler, (p, q)-Genocchi, (p, q)-Bernoulli, (p, q)-Frobenius-Euler, were firstly considered and developed by Duran et al. [2,3]; then, many authors worked on other (p, q)-special polynomials (see [6,8,10,12]). For instance, recently, Khan et al. [6] introduced (p, q)-Fubini-type polynomials and analyzed some of their basic properties. Obad et al. [8] defined and investigated *2D* (p, q)-Appell type polynomials in terms of determinantal aspect, and they provided several interesting properties. Sadjang [10] defined (p, q)-Appell type polynomials and provided some of their characterizations, including several algebraic properties. Sadjang et al. [12] introduced (p, q)-generalizations of two bivariate kinds of Bernoulli numbers and polynomials and then analyzed multifarious relations and formulae, including connection formulas, recurrence formulas, (p, q)-integral representations, and partial (p, q)-differential equations.

Special polynomials have important roles in several subjects of mathematics, engineering, and theoretical physics. The problems arising in mathematics, engineering and mathematical physics are framed in terms of differential equations. Most of these equations can only be treated by utilizing diverse families of special polynomials that give novel viewpoints of mathematical analysis. Moreover, they are widely used in computational models of engineering and scientific problems. In mathematics, these special polynomials yield the derivation of other useful identities in a fairly straightforward way and help to consider new families of special polynomials. Fubini-type polynomials appear in combinatorial mathematics and play an important role in the theory and applications of mathematics; hence, many number theory and combinatorics experts have extensively studied their properties and obtained a series of interesting results (see [6,13,14]). In addition, it is important that any polynomial has explicit formulas, symmetric identities, summation formulas, and relations with other polynomials. In this paper, our main aim is to consider (p,q)-sine and (p,q)-cosine Fubini polynomials and derive some of their properties and relations using series manipulation methods. The results derived in this work extend many earlier results for the several extensions of Fubini polynomials.

In this work, we make use of the following notations:

$$\mathbb{N}=\{1,2,3,\cdots\},\ \mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mathbb{Z}=\{\cdots,-2,-1,0,1,2,\cdots\} \text{ and } \mathbb{R}=(-\infty,\infty).$$

The (p,q)-numbers $[m]_{p,q}$ are defined as follows:

$$[m]_{p,q}=\frac{p^m-q^m}{p-q},\ 0<|q|<|p|\leq 1.$$

These can be rewritten such that $[m]_{p,q}=p^{m-1}[m]_{q/p}$, where $[m]_{q/p}$ is the q-number in quantum calculus (q-calculus) defined as $[m]_{q/p}=\frac{(q/p)^m-1}{(q/p)-1}$. Hence, it is observed that (p,q)-numbers and q-numbers are different; namely, one cannot derive (p,q)-numbers just by changing q by q/p in the definition of q-numbers. Again, when $p=1$, the (p,q)-numbers reduce to the q-numbers (see [4,5,9,11]).

The (p,q)-extension of the derivative operator of a function g with respect to t is given by

$$D_{p,q}g(t)=D_{p,q;t}g(t)=\frac{g(pt)-g(qt)}{(p-q)t}\quad (t\neq 0), \tag{1}$$

and $(D_{p,q}g(0))=g^{'}(0)$, provided that g is differentiable at 0. This operator satisfies the following properties

$$D_{p,q}(g(t)f(t))=f(qt)D_{p,q}g(t)+g(p(t))D_{p,q}f(t) \tag{2}$$

and

$$D_{p,q}\left(\frac{g(t)}{f(t)}\right)=\frac{f(qt)D_{p,q}g(t)-g(qt)D_{p,q}f(t)}{f(qt)f(pt)}. \tag{3}$$

The (p,q)-factorial numbers $[m]_{p,q}!$ and the(p,q)-binomial coefficients $\binom{m}{r}_{p,q}$ are provided by

$$[m]_{p,q}!=[m]_{p,q}\cdots[2]_{p,q}[1]_{p,q} \text{ for } m\in\mathbb{N} \text{ with } [0]_{p,q}=1$$

and

$$\binom{m}{r}_{p,q}=\frac{[m]_{p,q}!}{[r]_{p,q}![m-r]_{p,q}!}\quad (m\geq r).$$

The (p,q)-power basis is defined by

$$(t+a)(pt+aq)\cdots(p^{m-2}t+aq^{m-2})(p^{m-1}t+aq^{m-1})\ =\left(t\oplus_{p,q}a\right)^m\quad (m\geq 1)$$

and also has the following expansion

$$\sum_{r=0}^{m}\binom{m}{r}_{p,q}p^{\binom{m}{2}}q^{\binom{m-r}{2}}t^r a^{m-r}=\left(t\oplus_{p,q}a\right)^m. \tag{4}$$

The (p,q)-exponential functions, $e_{p,q}(t)$ and $E_{p,q}(t)$, are introduced by

$$e_{p,q}(t)=\sum_{m=0}^{\infty}\frac{p^{\binom{m}{2}}t^m}{[m]_{p,q}!} \text{ and } E_{p,q}(t)=\sum_{m=0}^{\infty}\frac{q^{\binom{m}{2}}t^m}{[m]_{p,q}!}, \tag{5}$$

which have the following relationships

$$e_{p^{-1}q^{-1}}(t) = E_{p,q}(t) \text{ and } e_{p,q}(t)E_{p,q}(-t) = 1. \tag{6}$$

These functions hold the following properties

$$D_{p,q}e_{p,q}(t) = e_{p,q}(pt) \text{ and } D_{p,q}E_{p,q}(t) = E_{p,q}(qt). \tag{7}$$

The (p,q)-analog of the usual definite integral is defined [9] by

$$\int_0^a g(t)d_{p,q}t = (p-q)a\sum_{r=0}^{\infty}\frac{p^r}{q^{r+1}}g\left(a\frac{p^r}{q^{r+1}}\right)$$

in conjunction with

$$\int_a^b g(t)d_{p,q}t = \int_0^b g(t)d_{p,q}t - \int_0^a g(t)d_{p,q}t. \tag{8}$$

From (5), it is observed that

$$e_{p,q}(it) = \sum_{m=0}^{\infty}\frac{p^{\binom{m}{2}}(it^m)}{[m]_{p,q}!} = \sum_{m=0}^{\infty}\frac{(-1^m)p^{\binom{2m}{2}}t^{2m}}{[m]_{p,q}!} + \sum_{m=0}^{\infty}\frac{(-1^m)p^{\binom{2m+1}{2}}t^{2m+1}}{[2m+1]_{p,q}!}. \tag{9}$$

From (9), the (p,q)-sine function and the (p,q)-cosine function are given [12] by

$$\sum_{m=0}^{\infty}\frac{(-1)^m p^{\binom{2m+1}{2}}t^{2m+1}}{[2m+1]_{p,q}!} = \sin_{p,q}(t) \text{ and } \sum_{m=0}^{\infty}\frac{(-1)^m p^{\binom{m}{2}}t^{2m}}{[2m]_{p,q}!} = \cos_{p,q}(t). \tag{10}$$

The (p,q)-Bernoulli, (p,q)-Euler, and (p,q)-Genocchi polynomials are introduced as follows (see [3]):

$$\frac{ze_{p,q}(tz)}{e_{p,q}(z)-1} = \sum_{m=0}^{\infty}B_m(t:p,q)\frac{z^m}{[m]_{p,q}!} \text{ for } |z|<2\pi, \tag{11}$$

$$\frac{[2]_{p,q}e_{p,q}(tz)}{e_{p,q}(z)+1} = \sum_{m=0}^{\infty}E_m(t:p,q)\frac{z^m}{[m]_{p,q}!} \text{ for } |z|<\pi \tag{12}$$

and

$$\frac{[2]_{p,q}ze_{p,q}(tz)}{e_{p,q}(z)+1} = \sum_{m=0}^{\infty}G_m(t:p,q)\frac{z^m}{[m]_{p,q}!} \text{ for } |z|<\pi. \tag{13}$$

When $t=0$, the polynomials given above reduce to their corresponding numbers, shown respectively by $B_m(p,q)$, $E_m(p,q)$, and $G_m(p,q)$ for $m\in\mathbb{N}_0$.

The generating function of geometric polynomials (or Fubini polynomials) is provided as follows (see [13,14]):

$$\frac{1}{1-t(e^z-1)} = \sum_{m=0}^{\infty}F_m(t)\frac{z^m}{m!}, \tag{14}$$

which implies

$$F_m(t) = \sum_{r=0}^{m}S_2(m,r)r!t^r, \tag{15}$$

where the numbers $S_2(m,r)$ are the Stirling numbers of the second kind provided by (see [15,16])

$$\frac{(e^z-1)^r}{r!} = \sum_{m=0}^{\infty}S_2(m,r)\frac{z^m}{m!}.$$

Upon setting $t = 1$, we attain $F_m(1) := F_m$, which denotes the corresponding Fubini numbers.

2. On (p,q)-Sine and (p,q)-Cosine Fubini Polynomials

The Taylor series expansions of the functions $e^{tz}\sin(wz)$ and $e^{tz}\cos(wz)$ are presented as given below (see [17])

$$\sum_{m=0}^{\infty} S_m(t,w)\frac{z^m}{m!} = e^{tz}\sin wz \text{ and } \sum_{m=0}^{\infty} C_m(t,w)\frac{z^m}{m!} = e^{tz}\cos wz, \tag{16}$$

where

$$S_m(t,w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r+1} t^{m-2r-1} w^{2r+1} \text{ and } C_m(t,w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r} t^{m-2r} w^{2r}. \tag{17}$$

Note that the symbol $\lfloor . \rfloor$ is the greatest integer function.

In the recent studies, Sadjang and Duran [12] considered (p,q)-generalizations of $S_m(t,w)$ and $C_m(t,w)$:

$$\sum_{m=0}^{\infty} S_{m,p,q}(t,w)\frac{z^m}{[m]_{p,q}!} = \sin_{p,q}(wz)e_{p,q}(tz) \tag{18}$$

and

$$\sum_{m=0}^{\infty} C_{m,p,q}(t,w)\frac{z^m}{[m]_{p,q}!} = \cos_{p,q}(wz)e_{p,q}(tz), \tag{19}$$

where

$$S_{m,p,q}(t,w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r+1}_{p,q} p^{(4r^2-2rm)+\binom{m-1}{2}} t^{(m-2r-1)} w^{(2r+1)} \tag{20}$$

and

$$C_{m,p,q}(t,w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r}_{p,q} p^{\binom{m}{2}+2r(r-m)} t^{m-2r} w^{2r}. \tag{21}$$

Now, we give our main definition as follows.

Definition 1. *For $p,q \in \mathbb{C}$ in conjunction with $0 < |q| < |p| \leq 1$, the (p,q)-sine and (p,q)-cosine Fubini polynomials $F_m^{(s)}(t,w;\gamma:p,q)$ and $F_m^{(c)}(t,w;\gamma:p,q)$ are introduced by*

$$\frac{e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}\sin_{p,q}(wz) = \sum_{m=0}^{\infty} \mathcal{F}_m^{(s)}(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!} \tag{22}$$

and

$$\frac{e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}\cos_{p,q}(wz) = \sum_{m=0}^{\infty} \mathcal{F}_m^{(c)}(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}. \tag{23}$$

Letting $w = 0$ in (22) and (23), we obtain the classical bivariate (p,q)-Fubini polynomials $F_m(t;\gamma:p,q)$ given by (*cf.* [6])

$$\frac{e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} = \sum_{m=0}^{\infty} F_m(t;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}.$$

Upon setting $w = 0$ and $t = 0$ in (22) and (23), we get the usual (p,q)-Fubini polynomials $F_m(\gamma : p,q)$ given by

$$\frac{1}{1-\gamma(e_{p,q}(z)-1)} = \sum_{m=0}^{\infty} F_m(\gamma : p,q) \frac{z^m}{[m]_{p,q}!}$$

and setting $t = w = 0$ and $\gamma = 1$ in (22) and (23), we obtain the familiar (p,q)-Fubini numbers $F_m(p,q)$ given by (*cf.* [6])

$$\frac{1}{2-e_{p,q}(z)} = \sum_{m=0}^{\infty} F_m(p,q) \frac{z^m}{[m]_{p,q}!}.$$

We state the following results.

Theorem 1. *The following summation formulae*

$$\sum_{r=0}^{m} \binom{m}{r}_{p,q} F_{r,p,q}(\gamma) S_{m-r,p,q}(t,w) = \mathcal{F}_m^{(s)}(t,w;\gamma : p,q) \tag{24}$$

and

$$\sum_{r=0}^{m} \binom{m}{r}_{p,q} F_{r,p,q}(\gamma) C_{m-r,p,q}(t,w) = \mathcal{F}_m^{(c)}(t,w;\gamma : p,q) \tag{25}$$

hold for $q,p \in \mathbb{C}$ in conjunction with $0 < |q| < |p| \leq 1$.

Proof. By (22) and (23), utilizing (18) and (19), we readily see that

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{F}_m^{(s)}(t,w;\gamma : p,q) \frac{z^m}{[m]_{p,q}!} &= \frac{\sin_{p,q}(wz) e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} \\
&= \left(\sum_{m=0}^{\infty} S_{m,p,q}(t,w) \frac{z^m}{[m]_{p,q}!} \right) \left(\sum_{m=0}^{\infty} F_m(p,q) \frac{z^m}{[m]_{p,q}!} \right) \\
&= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} \binom{m}{r}_{p,q} S_{m-r,p,q}(t,w) F_{r,p,q}(\gamma) \right) \frac{z^m}{[m]_{p,q}!}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{F}_m^{(c)}(t,w;\gamma : p,q) \frac{z^m}{[m]_{p,q}!} &= \frac{\cos_{p,q}(wz) e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} \\
&= \left(\sum_{m=0}^{\infty} C_{m,p,q}(t,w) \frac{z^m}{[m]_{p,q}!} \right) \left(\sum_{m=0}^{\infty} F_m(p,q) \frac{z^m}{[m]_{p,q}!} \right) \\
&= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} \binom{m}{r}_{p,q} C_{m-r,p,q}(t,w) F_{r,p,q}(\gamma) \right) \frac{z^m}{[m]_{p,q}!},
\end{aligned}$$

which complete the proofs of (24) and (25). □

Theorem 2. *The following summation formulae*

$$\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1}_{p,q}^{k} F_{m-1-2k}(t;\gamma : p,q) w^{2k+1} (-1) p^{\binom{2k+1}{2}} = \mathcal{F}_m^{(s)}(t,w;\gamma : p,q) \tag{26}$$

and

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}_{p,q} F_{m-2k}(t;\gamma : p,q) w^{2k} (-1)^k p^{\binom{2k}{2}} = \mathcal{F}_m^{(c)}(t,w;\gamma : p,q) \tag{27}$$

are valid for $m \geq 0$ and $p,q \in \mathbb{C}$ in conjunction with $0 < |q| < |p| \leq 1$.

Proof. By (22) and (23), using (10), we observe that

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{F}_m^{(s)}(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!} &= \frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} \\
&= \left(\sum_{m=0}^{\infty} \frac{(-1)^m p^{\binom{2m+1}{2}}(wz)^{2m+1}}{[2m+1]_{p,q}!}\right)\left(\sum_{m=0}^{\infty} F_m^{(s)}(t;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right) \\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}(-1)^k\binom{m}{2k+1}_{p,q} p^{\binom{2k+1}{2}}F_{m-1-2k}(t;\gamma:p,q)w^{2k+1}\right)\frac{z^m}{[m]_{p,q}!}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{F}_m^{(c)}(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!} &= \frac{\cos_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} \\
&= \left(\sum_{m=0}^{\infty} \frac{(-1)^m p^{\binom{m}{2}}t^{2m}}{[2m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty} F_m^{(s)}(t;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right) \\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}F_{m-2k}(t;\gamma:p,q)(-1)^k\binom{m}{2k}_{p,q} p^{\binom{2k}{2}}w^{2k}\right)\frac{z^m}{[m]_{p,q}!},
\end{aligned}$$

which means the asserted results (26) and (27). □

Theorem 3. *The following relationships*

$$\gamma\sum_{r=0}^{m} p^{\binom{r}{2}+\binom{m-r}{2}}\binom{m}{r}_{p,q}\mathcal{F}_r^{(s)}(t,w;\gamma:p,q)+S_{m,p,q}(t,w) = (\gamma+1)\mathcal{F}_m^{(s)}(t,w;\gamma:p,q) \tag{28}$$

and

$$\gamma\sum_{r=0}^{m} p^{\binom{r}{2}+\binom{m-r}{2}}\binom{m}{r}_{p,q}\mathcal{F}_r^{(c)}(t,w;\gamma:p,q)+C_{m,p,q}(t,w) = (\gamma+1)\mathcal{F}_m^{(c)}(t,w;\gamma:p,q) \tag{29}$$

hold for $m \geq 0$ and $q,p \in \mathbb{C}$ in conjunction with $0 < |q| < |p| \leq 1$.

Proof. Utilizing (18), (19), (22) and (23), the proofs of (28) and (29) are based on the following equalities:

$$\frac{\gamma+1-\gamma e_{p,q}(z)}{1-\gamma(e_{p,q}(z)-1)}\sin_{p,q}(wz)e_{p,q}(tz) = \sin_{p,q}(wz)e_{p,q}(tz)$$

and

$$\frac{\gamma+1-\gamma e_{p,q}(z)}{1-\gamma(e_{p,q}(z)-1)}\cos_{p,q}(wz)e_{p,q}(tz) = \cos_{p,q}(wz)e_{p,q}(tz).$$

Therefore, we omit the details of the proofs. □

Theorem 4. *The following formulae*

$$\sum_{r=0}^{m}(t_1^r+t_2^r)\binom{m}{r}_{p,q}\mathcal{F}_{m-r}^{(s)}(0,w;\gamma:p,q)p^{\binom{r}{2}} = \mathcal{F}_m^{(s)}(t_1,w;\gamma:p,q)+\mathcal{F}_m^{(s)}(t_1,w;\gamma:p,q) \tag{30}$$

and

$$\sum_{r=0}^{m}(t_1^r+t_2^r)\binom{m}{r}_{p,q}\mathcal{F}_{m-r}^{(c)}(0,w;\gamma:p,q)p^{\binom{r}{2}}=\mathcal{F}_m^{(c)}(t_1,w;\gamma:p,q)+\mathcal{F}_m^{(c)}(t_1,w;\gamma:p,q) \tag{31}$$

are valid for $m\geq 0$ and $p,q\in\mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. Utilizing (18), (19), (22) and (23), the proofs of (28) and (29) are based on the following equalities:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left[\mathcal{F}_m^{(s)}(t_1,w;\gamma:p,q)+\mathcal{F}_m^{(s)}(t_2,w;\gamma:p,q)\right]\frac{z^m}{[m]_{p,q}!}\\
&=\left(\sum_{m=0}^{\infty}p^{\binom{m}{2}}(t_1^m+t_2^m)\frac{z^m}{[m]_{p,q}!}\right)\left(\frac{\sin_{p,q}(wz)}{1-\gamma(e_{p,q}(z)-1)}\right)\\
&=\sum_{m=0}^{\infty}\left(\sum_{r=0}^{m}\binom{m}{r}_{p,q}\mathcal{F}_{m-r}^{(s)}(0,w;\gamma:p,q)(t_1^r+t_2^r)p^{\binom{r}{2}}\right)\frac{z^m}{[m]_{p,q}!}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left[\mathcal{F}_m^{(c)}(t_1,w;\gamma:p,q)+\mathcal{F}_m^{(c)}(t_2,w;\gamma:p,q)\right]\frac{z^m}{[m]_{p,q}!}\\
&=\left(\sum_{m=0}^{\infty}p^{\binom{m}{2}}(t_1^m+t_2^m)\frac{z^m}{[m]_{p,q}!}\right)\left(\frac{\cos_{p,q}(wz)}{1-\gamma(e_{p,q}(z)-1)}\right)\\
&=\sum_{m=0}^{\infty}\left(\sum_{r=0}^{m}\binom{m}{r}_{p,q}\mathcal{F}_{m-r}^{(c)}(0,w;\gamma:p,q)(t_1^r+t_2^r)p^{\binom{r}{2}}\right)\frac{z^m}{[m]_{p,q}!}.
\end{aligned}$$

Therefore, we omit the details of the proofs. □

Now, we give (p,q)-derivative operator rules and (p,q)-integral representations for the (p,q)-sine and (p,q)-cosine Fubini polynomials with the following theorems.

Theorem 5. *The following derivate formulae*

$$\begin{aligned}
\frac{\partial}{\partial_{p,q}t}\mathcal{F}_m^{(s)}(t,w;\gamma:p,q)&=\mathcal{F}_{m-1}^{(s)}(pt,w;\gamma:p,q)[m]_{p,q}\\
\frac{\partial}{\partial_{p,q}w}\mathcal{F}_m^{(s)}(t,w;\gamma:p,q)&=\mathcal{F}_{m-1}^{(s)}(t,qw;\gamma:p,q)[m]_{p,q}\\
\frac{\partial}{\partial_{p,q}t}\mathcal{F}_m^{(c)}(t,w;\gamma:p,q)&=\mathcal{F}_{m-1}^{(c)}(pt,w;\gamma:p,q)[m]_{p,q}\\
\frac{\partial}{\partial_{p,q}w}\mathcal{F}_m^{(c)}(t,w;\gamma:p,q)&=\mathcal{F}_{m-1}^{(c)}(t,qw;\gamma:p,q)[m]_{p,q}
\end{aligned} \tag{32}$$

hold for $m\geq 0$ and $q,p\in\mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. If we apply the (p,q)-derivative operator to the exponential generating function (22) with respect to t, by utilizing (7), we see that

$$\sum_{m=0}^{\infty}\frac{\partial}{\partial_{p,q}t}\mathcal{F}_m^{(s)}(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}=\frac{\sin_{p,q}(wz)\frac{\partial}{\partial_{p,q}t}e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}$$
$$=z\frac{\sin_{p,q}(wz)e_{p,q}(ptz)}{1-\gamma(e_{p,q}(z)-1)}$$
$$=\sum_{m=0}^{\infty}\mathcal{F}_m^{(s)}(pt,w;\gamma:p,q)\frac{z^{m+1}}{[m]_{p,q}!},$$

which implies (32). The others can be readily proved similarly. □

Theorem 6. *The following (p,q)-integral representations*

$$\int_{\varrho}^{v}\mathcal{F}_m^{(s)}(t,w;\gamma:p,q)d_{p,q}t=\frac{\mathcal{F}_{m+1}^{(s)}(\frac{v}{p},w;\gamma:p,q)-\mathcal{F}_{m+1}^{(s)}(\frac{\varrho}{p},w;\gamma:p,q)}{[m+1]_{p,q}}$$

and

$$\int_{\varrho}^{b}\mathcal{F}_m^{(c)}(t,w;\gamma:p,q)d_{p,q}t=\frac{\mathcal{F}_{m+1}^{(c)}(\frac{v}{p},w;\gamma:p,q)-\mathcal{F}_{m+1}^{(c)}(\frac{\varrho}{p},w;\gamma:p,q)}{[m+1]_{p,q}}$$

are valid for $m\geq 0$ and $q,p\in\mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. Since

$$\int_{\varrho}^{v}\frac{\partial g(t)}{\partial_{p,q}t}d_{p,q}t=g(v)-g(\varrho),$$

(see [9]), using Theorem 5, (22) and (23), we investigate

$$\int_{\varrho}^{v}\frac{\partial}{\partial_{p,q}t}\mathcal{F}_m^{(s)}(t,w;\gamma:p,q)d_{p,q}t=\frac{1}{[m+1]_{p,q}}\int_{\varrho}^{v}\mathcal{F}_{m+1}^{(s)}(\frac{t}{p},w;\gamma:p,q)d_{p,q}t$$
$$=\frac{\mathcal{F}_{m+1}^{(s)}(\frac{v}{p},w;\gamma:p,q)-\mathcal{F}_{m+1}^{(s)}(\frac{\varrho}{p},w;\gamma:p,q)}{[m+1]_{p,q}}$$

and

$$\int_{\varrho}^{v}\frac{\partial}{\partial_{p,q}t}\mathcal{F}_m^{(c)}(t,w;\gamma:p,q)d_{p,q}t=\frac{1}{[m+1]_{p,q}}\int_{\varrho}^{v}\mathcal{F}_{m+1}^{(c)}(\frac{t}{p},w;\gamma:p,q)d_{p,q}t$$
$$=\frac{\mathcal{F}_{m+1}^{(c)}(\frac{v}{p},w;\gamma:p,q)-\mathcal{F}_{m+1}^{(c)}(\frac{\varrho}{p},w;\gamma:p,q)}{[m+1]_{p,q}},$$

which completes the proof of the theorem. □

Now, we state the following summation formula.

Theorem 7. *The following summation formulae*

$$\frac{\gamma_2\mathcal{F}_m^{(s)}(t,w;\gamma_1:p,q)-\gamma_1\mathcal{F}_m^{(s)}(t,w;\gamma_2:p,q)}{\gamma_2-\gamma_1}=\sum_{r=0}^{m}\binom{m}{r}_{p,q}\mathcal{F}_{m-r}^{(s)}(t,w;\gamma_1:p,q)\mathcal{F}_r^{(s)}(\gamma_2:p,q)\tag{33}$$

and

$$\frac{\gamma_2\mathcal{F}_m^{(c)}(t,w;\gamma_1:p,q)-\gamma_1\mathcal{F}_m^{(c)}(t,w;\gamma_2:p,q)}{\gamma_2-\gamma_1}=\sum_{r=0}^{m}\binom{m}{r}_{p,q}\mathcal{F}_{m-r}^{(c)}(t,w;\gamma_1:p,q)\mathcal{F}_r^{(c)}(\gamma_2:p,q)\tag{34}$$

hold for $\gamma_1\neq\gamma_2$, $m\geq 0$ and $q,p\in\mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. By (22) and (23), we observe that

$$\begin{aligned}
&\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{(1-\gamma_2(e_{p,q}(z)-1))(1-\gamma_1(e_{p,q}(z)-1))} \\
&= \frac{\gamma_2}{\gamma_2-\gamma_1}\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma_1(e_{p,q}(z)-1)} - \frac{e\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma_2(e_{p,q}(z)-1)}\frac{\gamma_1}{\gamma_2-\gamma_1} \\
&= \sum_{m=0}^{\infty}\left(\frac{\gamma_2\mathcal{F}_m^{(s)}(t,w;\gamma_1:p,q)-\gamma_1\mathcal{F}_m^{(s)}(t,w;\gamma_2:p,q)}{\gamma_2-\gamma_1}\right)\frac{z^m}{[m]_{p,q}!}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\cos_{p,q}(wz)e_{p,q}(tz)}{(1-\gamma_1(e_{p,q}(z)-1))(1-\gamma_2(e_{p,q}(z)-1))} \\
&= \frac{\gamma_2}{\gamma_2-\gamma_1}\frac{\cos_{p,q}(wz)e_{p,q}(tz)}{1-\gamma_1(e_{p,q}(z)-1)} - \frac{\gamma_1}{\gamma_2-\gamma_1}\frac{\cos_{p,q}(wz)e_{p,q}(tz)}{1-\gamma_2(e_{p,q}(z)-1)} \\
&= \sum_{m=0}^{\infty}\left(\frac{\gamma_2\mathcal{F}_m^{(c)}(t,w;\gamma_1:p,q)-\gamma_1\mathcal{F}_m^{(c)}(t,w;\gamma_2:p,q)}{\gamma_2-\gamma_1}\right)\frac{z^m}{[m]_{p,q}!},
\end{aligned}$$

which means the claimed results (33) and (34). □

Here are summation formulae for the (p,q)-sine Fubini polynomials and (p,q)-cosine Fubini polynomials.

Theorem 8. *The following formulae*

$$\sum_{r=0}^{m} p^{\binom{m-r}{2}}\binom{m}{r}_{p,q}\mathcal{F}_r^{(s)}(t,w;\gamma:p,q)+\frac{(t\oplus w)_{p,q}^m}{\gamma}=(1+\gamma)\mathcal{F}_m^{(s)}(t,w;\gamma:p,q) \tag{35}$$

and

$$\sum_{r=0}^{m} p^{\binom{m-r}{2}}\binom{m}{r}_{p,q}\mathcal{F}_r^{(c)}(t,w;\gamma:p,q)+\frac{(t\oplus w)_{p,q}^m}{\gamma}=(1+\gamma)\mathcal{F}_m^{(c)}(t,w;\gamma:p,q) \tag{36}$$

are valid for $m\geq 0$ and $p,q\in\mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. Utilizing the following equality

$$\frac{1+\gamma}{(1-\gamma(e_{p,q}(z)-1))\gamma e_{p,q}(z)}=\frac{1}{1-\gamma(e_{p,q}(z)-1)}+\frac{1}{\gamma e_{p,q}(z)}$$

and from (22) and (23), we acquire

$$\frac{(1+\gamma)e_{p,q}(tz)\sin_{p,q}(wz)}{(1-\gamma(e_{p,q}(z)-1))\gamma e_{p,q}(z)}=\frac{e_{p,q}(tz)\sin_{p,q}(wz)}{1-\gamma(e_{p,q}(z)-1)}+\frac{e_{p,q}(tz)\sin_{p,q}(wz)}{\gamma e_{p,q}(z)}$$

and

$$\frac{(1+\gamma)e_{p,q}(tz)\cos_{p,q}(wz)}{(1-\gamma(e_{p,q}(z)-1))\gamma e_{p,q}(z)}=\frac{e_{p,q}(tz)\cos_{p,q}(wz)}{1-\gamma(e_{p,q}(z)-1)}+\frac{e_{p,q}(tz)\cos_{p,q}(wz)}{\gamma e_{p,q}(z)},$$

which give the claimed results (35) and (36). □

Now, we derive some correlations for the (p,q)-sine and (p,q)-cosine Fubini polynomials in (22) and (23) associated with the (p,q)-Bernoulli, Euler, and Genocchi polynomials (11)–(13) and the (p,q)-Stirling numbers of the second kind. We first provide the following theorem.

Theorem 9. *Each of the following correlations*

$$\sum_{r=0}^{m+1}\binom{m+1}{r}_{p,q}\left(\frac{B_r(1:p,q)-B_r(p,q)}{[m+1]_{p,q}}\right)\mathcal{F}^{(s)}_{m+1-r}(t,w;\gamma:p,q)=\mathcal{F}^{(s)}_{m}(t,w;\gamma:p,q) \quad (37)$$

and

$$\sum_{r=0}^{m+1}\binom{m+1}{r}_{p,q}\left(\frac{B_r(1:p,q)-B_r(p,q)}{[m+1]_{p,q}}\right)\mathcal{F}^{(c)}_{m+1-r}(t,w;\gamma:p,q)=\mathcal{F}^{(c)}_{m}(t,w;\gamma:p,q) \quad (38)$$

are valid for $m \geq 0$ and $p,q \in \mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. From (11) and (22), we consider that

$$\begin{aligned}\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} &= \frac{e_{p,q}(z)-1}{z}\frac{z}{e_{p,q}(z)-1}\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}\\ &=\frac{1}{z}\left(\sum_{m=0}^{\infty}B_m(1:p,q)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty}\mathcal{F}^{(s)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right)\\ &\quad-\frac{1}{z}\left(\sum_{m=0}^{\infty}B_m(p,q)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty}\mathcal{F}^{(s)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right),\end{aligned}$$

which means the claimed correlation (37). The proof of the other correlation (38) can be done similarly to the proof of the correlation (37). □

Theorem 10. *The following correlations*

$$\sum_{r=0}^{m}\binom{m}{r}_{p,q}\left(\frac{E_r(1:p,q)+E_r(p,q)}{[2]_{p,q}}\right)\mathcal{F}^{(s)}_{m-r}(t,w;\gamma:p,q)=\mathcal{F}^{(s)}_{m}(t,w;\gamma:p,q) \quad (39)$$

and

$$\sum_{r=0}^{m}\binom{m}{r}_{p,q}\left(\frac{E_r(1:p,q)+E_r(p,q)}{[2]_{p,q}}\right)\mathcal{F}^{(c)}_{m-r}(t,w;\gamma:p,q)=\mathcal{F}^{(c)}_{m}(t,w;\gamma:p,q) \quad (40)$$

hold for $m \geq 0$ and $q,p \in \mathbb{C}$ in conjunction with $0<|q|<|p|\leq 1$.

Proof. By (12) and (23), we see that

$$\begin{aligned}\frac{\cos_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)} &= \frac{e_{p,q}(z)+1}{[2]_{p,q}}\frac{[2]_{p,q}}{e_{p,q}(z)+1}\frac{\cos_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}\\ &=\frac{1}{[2]_{p,q}}\left(\sum_{m=0}^{\infty}E_m(1:p,q)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty}\mathcal{F}^{(c)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right)\\ &\quad+\frac{1}{[2]_{p,q}}\left(\sum_{m=0}^{\infty}E_m(p,q)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty}\mathcal{F}^{(c)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right),\end{aligned}$$

which implies the asserted correlation (40). The proof of the other correlation (39) can be done similarly to the proof of the correlation (40). □

Theorem 11. *Each of the following correlations*

$$\sum_{r=0}^{m+1}\binom{m+1}{r}_{p,q}\left(\frac{G_r(1:p,q)+G_r(p,q)}{[2]_{p,q}[m+1]_{p,q}}\right)\mathcal{F}^{(s)}_{m+1-r}(t,w;\gamma:p,q)=\mathcal{F}^{(s)}_{m}(t,w;\gamma:p,q) \quad (41)$$

and

$$\sum_{r=0}^{m+1}\binom{m+1}{r}_{p,q}\left(\frac{G_r(1:p,q)+G_r(p,q)}{[2]_{p,q}[m+1]_{p,q}}\right)\mathcal{F}^{(c)}_{m+1-r}(t,w;\gamma:p,q)=\mathcal{F}^{(c)}_m(t,w;\gamma:p,q) \quad (42)$$

are valid for $m\geq 0$ *and* $q,p\in\mathbb{C}$ *in conjunction with* $0<|q|<|p|\leq 1$.

Proof. From (13) and (22), we investigate

$$\begin{aligned}\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}&=\frac{e_{p,q}(z)+1}{z[2]_{p,q}}\frac{z[2]_{p,q}}{e_{p,q}(z)+1}\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}\\&=\frac{1}{z[2]_{p,q}}\left(\sum_{m=0}^{\infty}\mathcal{F}^{(s)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty}G_m(1:p,q)\frac{z^m}{[m]_{p,q}!}\right)\\&+\frac{1}{z[2]_{p,q}}\left(\sum_{m=0}^{\infty}\mathcal{F}^{(s)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{m=0}^{\infty}G_m(p,q)\frac{z^m}{[m]_{p,q}!}\right),\end{aligned}$$

which means the desired correlation (41). The proof of the other correlation (42) can be done similarly to the proof of the correlation (41). □

The (p,q)-Stirling numbers $S_2(m,r:p,q)$ of the second kind are defined by (*cf.* [2])

$$\frac{(e_{p,q}(z)-1)^r}{[r]_{p,q}!}=\sum_{m=r}^{\infty}S_2(m,r:p,q)\frac{z^m}{[m]_{p,q}!}.$$

Theorem 12. *The following correlations*

$$\sum_{k=0}^{m}\sum_{r=0}^{k})\gamma^r[r]_{p,q}!S_{m-k,p,q}(t,w)\binom{m}{k}_{p,q}S_2(k,r:p,q=\mathcal{F}^{(s)}_m(t,w;\gamma:p,q) \quad (43)$$

and

$$\sum_{k=0}^{m}\sum_{r=0}^{k}\gamma^r[r]_{p,q}!C_{m-k,p,q}(t,w)\binom{m}{k}_{p,q}S_2(k,r:p,q)=\mathcal{F}^{(c)}_m(t,w;\gamma:p,q) \quad (44)$$

hold for $m\geq 0$ *and* $q,p\in\mathbb{C}$ *in conjunction with* $0<|q|<|p|\leq 1$.

Proof. From (22), using the series manipulation method, we attain

$$\begin{aligned}\sum_{m=0}^{\infty}\mathcal{F}^{(s)}_m(t,w;\gamma:p,q)\frac{z^m}{[m]_{p,q}!}&=\frac{\sin_{p,q}(wz)e_{p,q}(tz)}{1-\gamma(e_{p,q}(z)-1)}\\&=\sin_{p,q}(wz)e_{p,q}(tz)\sum_{r=0}^{\infty}\gamma^r(e_{p,q}(z)-1)^r\\&=\left(\sum_{m=0}^{\infty}S_{m,p,q}(t,w)\frac{z^m}{[m]_{p,q}!}\right)\left(\sum_{r=0}^{\infty}\gamma^r\sum_{m=r}^{\infty}[r]_{p,q}!S_2(m,r:p,q)\frac{z^m}{[m]_{p,q}!}\right)\\&=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{r=0}^{k}\gamma^r[r]_{p,q}!S_{m-k,p,q}(t,w)\binom{m}{k}_{p,q}S_2(k,r:p,q)\right)\frac{z^m}{[m]_{p,q}!},\end{aligned}$$

which proves the correlation (43). The proof of the other correlation (44) can be done similarly to the proof of the correlation (43). □

3. Conclusions

In recent years, (p,q)-extensions of many special polynomials. such as Bernoulli, Euler, Genocchi, and Hermite polynomials, have been considered and investigated by many mathematicians (see [2,3,6,8,10,12]).

In this work, (p,q)-sine Fubini polynomials and (p,q)-cosine Fubini polynomials have been introduced and multifarious summation formulae and relationships for these polynomials have been derived by utilizing some series manipulation methods. Furthermore, (p,q)-derivative operator rules and (p,q)-integral representations for the (p,q)-sine Fubini polynomials and (p,q)-cosine Fubini polynomials have been provided. Moreover, diverse correlations related to both the (p,q)-Stirling numbers and the (p,q)-Euler, Bernoulli, and Genocchi polynomials have been developed. When $q \to p = 1$, all acquired results in this work reduce to classical results for sine-Fubini and cosine-Fubini polynomials. The results obtained in this paper are also generalizations of the many earlier (p,q)-results, some of which involve related references in [6].

We think that this idea of constructing new (p,q)-polynomial sequences has possible applications in physics, science, and engineering, as well as in mathematics, such as in combinatorics, integral transforms, approximation theory, and analytic number theory; see [1–12] and the references cited therein. As one of our future research projects, we would like to continue and extend this idea in various directions.

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References

1. Corcino, R.B. On (p,q)-binomial coefficients. *Electron. J. Combin. Number Theory* **2008**, *8*, A29.
2. Duran, U.; Acikgoz, M. Apostol type (p,q) -Frobenious-Euler polynomials and numbers. *Kragujevac J. Math.* **2018**, *42*, 555–567. [CrossRef]
3. Duran, U.; Acikgoz, M.; Araci, S. On (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials. *J. Comput. Theor. Nanosci.* **2016**, *13*, 7833–7908. [CrossRef]
4. Gupta, V. (p,q)-Baskakov-Kontorovich operators. *Appl. Math. Inf. Sci.* **2016**, *10*, 1551–1556. [CrossRef]
5. Jain, P.; Basu, C.; Panwar, V. On the (p,q)-Mellin transform and its applications. *Acta Math Sci.* **2021**, *4*, 1719–1732. [CrossRef]
6. Khan, W.A.; Nisar, K.S.; Baleanu, D. A note on (p,q) -analogue type of Fubini numbers and polynomials. *Aims Math.* **2020**, *5*, 2743–2757. [CrossRef]
7. Milovanovic, G.V.; Gupta, V.; Malik, N. (p,q)-Beta functions and applications in approximation. *Bol. Soc. Mat. Mex.* **2018**, *24*, 219–237. [CrossRef]
8. Obad, A.M.; Khan, A.; Mursaleen, M.; Yasmin, G. A determinantal approach to post quantum analogue of Appell type polynomials. *Int. J. Appl. Comput. Math.* **2022**, *8*, 15. [CrossRef]
9. Sadjang, P.N. On the fundamental theorem of (p,q) -calculus and some (p,q)-Taylor formulas. *Results Math.* **2018**, *73*, 39. [CrossRef]
10. Sadjang, P.N. On (p,q)-Appell polynomials. *Anal. Math.* 2018, *in press*.
11. Sadjang, P.N. On two (p,q)-analogues of the Laplace transform. *J. Diff. Equ. Appl.* **2018**, *23*, 1562–1583.
12. Sadjang, P.N.; Duran, U. On two bivariate kinds of (p,q)-Bernoulli polynomials. *Miskolc Math. Notes* **2019**, *20*, 1185–1199. [CrossRef]
13. Dil, A.; Kurt, V. Investigating geometric and exponential polynomials with Euler-Seidel matrices. *J. Integer Seq.* **2011**, *14*, 1–12.
14. Kargin, L. Some formulae for products of Fubini polynomials with applications. *arXiv* **2016**, arXiv:1701.01023.
15. Boyadzhiev, K.N. A series transformation formula and related polynomials. *Int. J. Math. Math. Sci.* **2005**, *23*, 3849–3866. [CrossRef]
16. Tanny, S.M. On some numbers related to Bell numbers. *Canad. Math. Bull.* **1974**, *17*, 733–738. [CrossRef]
17. Jamei, M.-M.; Koepf, W. Symbolic computation of some power-trigonometric series. *J. Symbolic Comput.* **2017**, *80*, 273–284. [CrossRef]