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On Gould–Hopper-Based Fully Degenerate Poly-Bernoulli Polynomials with a q-Parameter

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Abstract: We firstly consider the fully degenerate Gould–Hopper polynomials with a q parameter and investigate some of their properties including difference rule, inversion formula and addition formula. We then introduce the Gould–Hopper-based fully degenerate poly-Bernoulli polynomials with a q parameter and provide some of their diverse basic identities and properties including not only addition property, but also difference rule properties. By the same way of mentioned polynomials, we define the Gould–Hopper-based fully degenerate (α, q)-Stirling polynomials of the second kind, and then give many relations. Moreover, we derive multifarious correlations and identities for foregoing polynomials and numbers, including recurrence relations and implicit summation formulas.

Keywords: Gould–Hopper polynomials; Bernoulli polynomials; Hermite polynomials; poly Bernoulli polynomials; Stirling numbers of second kind; Polylogarithm functions; Cauchy product

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1. Introduction

Special functions possess a lot of importances in numerous fields of mathematics, physics, engineering and other related disciplines covering different topics such as differential equations, mathematical analysis, functional analysis, mathematical physics, quantum mechanics and so on. Particularly, the family of special polynomials is one of the most useful, widespread and applicable family of special functions. Some of the most considerable polynomials in the theory of special polynomials are Bernoulli polynomails (see [1,2]) and the generalized Hermite–Kampé de Fériet (or Gould–Hopper) polynomials (see [3]). Recently, aforementioned polynomials and their diverse extensions have been studied and developed by lots of physicsics and mathematicians, see [1,3–18] and references cited therein. Araci et al. [4] considered a novel concept of the Apostol Hermite-Genocchi polynomials by using the modified Milne–Thomson's polynomials and obtained several implicit summation formulae and general symmetric identities arising from different analytical means and generating functions method. Bretti et al. [6] gave multidimensional extensions of the Bernoulli and Appell polynomials by utilizing the Hermite–Kampé de Fériet polynomials and provided the differential equations, satisfing by the corresponding $2D$ polynomials, acquired from exploiting the factorization method. Bayad et al. [5] considered poly-Bernoulli polynomials and numbers and proved a collection of extremely important and fundamental identities satisfied by them. Cenkci et al. [7] handled poly-Bernoulli numbers and polynomials with a q parameter and investigated several aritmetical and number theoretical properties. Dattoli et al. [9] applied the method of generating function to define novel forms of Bernoulli numbers and polynomials, which were exploited to get further classes of partial sums including generalized numerous index many variable polynomials. Khan et al. [11,12] defined the Hermite poly-Bernoulli polynomials and numbers of the second kind and the degenerate Hermite poly-Bernoulli polynomials and numbers and analyzed many of their applications in combinatorics, number theory and other fields of mathematics. Kim et al. [13–15] dealt with the several degenerate poly-Bernoulli polynomials and numbers. Kurt et al. [16] studied on the Hermite–Kampé de Fériet based second kind Genocchi polynomials and presented diverse relationships for them. Ozarslan [19] introduced an unified family of Hermite-based Apostol–Bernoulli, Euler and Genocchi polynomials and then attained some symmetry identities between these polynomials and the generalized sum of integer powers. Ozarslan also provided explicit closed-form formulae for this unified family and proved a finite series relation between this unification and $3d$-Hermite polynomials. Pathan [20] presented a new class of generalized Hermite–Bernoulli polynomials and emerged multifarious implicit summation formulae and symmetric identities by using different analytical means appying generating functions. Pathan et al. [21] introduced a new class of generalized polynomials associated with the modified Milne–Thomson's polynomials $\Phi_n^{(\alpha)}(x,v)$ of degree n and order α and provided some of their properties.

In this paper, the usual notations $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers and the set of all nonnegative integers, respectively.

An outline of this paper is as follows. Section 2 covers the rudiments and some basic symbols and operators. Section 3 deals with the fully degenerate Gould–Hopper polynomials with a q parameter. Section 4 mainly analyzes the Gould–Hopper-based fully degenerate poly-Bernoulli polynomials with a q parameter and provides the several properties for these polynomials. Section 5 gives the definition of the Gould–Hopper-based fully degenerate (α, q)-Stirling numbers of the second kind and provides some relations for these numbers. Finally, we derive multifarious correlations and formulas including the fully degenerate Gould–Hopper polynomials with a q parameter, the Gould–Hopper-based fully degenerate poly-Bernoulli polynomials with a q parameter and the Gould–Hopper-based fully degenerate (α, q)-Stirling numbers of the second kind.

2. Preliminary Informations and Δ_ω Difference Operator

The Gould–Hopper family of polynomials is defined by the exponential generating function (see [6])

$$\sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!} = e^{xt+yt^j}, \tag{1}$$

where $j \in \mathbb{N}$ with $j \geq 2$. In the case $j = 1$, the corresponding bivariate polynomials are simply expressed by the Newton binomial formula. Upon setting $j = 2$ in (1) gives the classical Hermite polynomials $H_n^{(2)}(x,y)$ and the mentioned polynomials have been used to define bivariate extensions of some special polynomails, such as Bernoulli and Euler polynomials (see [9]).

For $k \in \mathbb{Z}$ with $k > 1$, the k-th polylogarithm function is defined by (see [5,7,10,17])

$$Li_k(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^k} \quad (t \in \mathbb{C} \text{ with } |t| < 1). \tag{2}$$

We always assume $|t| < 1$ along this paper. When $k = 1$, $Li_1(t) = -\log(1-t)$. In the case $k \leq 0$, $Li_k(t)$ are the rational functions:

$$Li_0(t) = \frac{t}{1-t}, \quad Li_{-1}(t) = \frac{t}{(1-t)^2}, \quad Li_{-2}(t) = \frac{t^2+t}{(1-t)^3}, \quad Li_{-3}(t) = \frac{t^3+4t^2+t}{(1-t)^4}, \cdots.$$

Now, let us recall some basic notations and definitions the reader should know.

Definition 1 (See [8,18]). *Let congomega be a non-zero complex number, the ω-falling factorial is defined by*

$$x^{(n,\omega)}=\begin{cases} x(x-\omega)(x-2\omega)\cdots(x-(n-1)\omega), & n=1,2,\ldots \\ 1 & n=0 \end{cases}$$

The ω-Pochhammer is defined by

$$(x)_{(n,\omega)}=\begin{cases} x(x+\omega)(x+2\omega)\cdots(x+(n-1)\omega), & n=1,2,\ldots \\ 1 & n=0 \end{cases}$$

When $\omega=1$, the ω-falling factorial is the usual falling factorial

$$x^{(n,1)}=x(x-1)\cdots(x-n+1)$$

and the ω-Pochhammer is the usual Pochhammer [2,22]

$$(x)_{(n,1)}=(x)_n=x(x+1)\cdots(x+n-1).$$

Note that the ω-falling factorial and the ω-Pochhammer are linked by the relation

$$x^{(n,\omega)}=(-1)^n(-x)_{(n,\omega)}.$$

Definition 2 (See [8,18]). *The Δ_ω difference operator is defined by*

$$\Delta_\omega f(x)=\frac{1}{\omega}(f(x+\omega)-f(x)),\quad \omega\neq 0. \tag{3}$$

Proposition 1. *The following difference rule holds true:*

$$\Delta_\omega{}^k x^{(n,\omega)}=\frac{n!}{(n-k)!}x^{(n-k,\omega)},\quad 0\leq k\leq n. \tag{4}$$

Proof. We prove the result for $k=1$, the general case is obtained by induction.

$$\begin{aligned}
\Delta_\omega x^{n,\omega} &= \frac{1}{\omega}\left(\prod_{j=0}^{n-1}(x+\omega-j\omega)-\prod_{j=0}^{n-1}(x-j\omega)\right)\\
&= \frac{1}{\omega}\left(\prod_{j=0}^{n-1}(x-(j-1)\omega)-\prod_{j=0}^{n-1}(x-j\omega)\right)\\
&= \frac{1}{\omega}\left((x+\omega)\prod_{j=0}^{n-2}(x-j\omega)-(x-(n-1)\omega)\prod_{j=0}^{n-2}(x-j\omega)\right)\\
&= \frac{1}{\omega}[(x-\omega)-(x-(n-1)\omega)]\prod_{j=0}^{n-2}(x-j\omega)\\
&= nx^{(n-1,\omega)}.
\end{aligned}$$

□

Proposition 2. *Let $f(x)$ be a polynomial of degree N, then the following Taylor formula holds true:*

$$f(x)=\sum_{k=0}^{N}\frac{(\Delta_\omega^k f)(0)}{k!}x^{(k,\omega)}. \tag{5}$$

Proof. Since $\{x^{(n,\omega)}\}_{n=0}^{\infty}$ forms a basis of the polynomial ring, there exist constants $a_0, \ldots, a_N$ such that

$$f(x) = \sum_{k=0}^{N} a_k x^{(k,\omega)}.$$

Applying Δ_ω j times on $f(x)$, we get

$$\Delta_\omega{}^j f(x) = \sum_{k=j}^{N} a_k \frac{k!}{(k-j)!} x^{(k-j,\omega)} = a_j j! + \sum_{k=j}^{N} a_k \frac{k!}{(k-j)!} x^{(k-j,\omega)}.$$

Thus $(\Delta_\omega{}^j f)(0) = a_j j!$ and the proposition follows. □

The following Lemma will be useful in the derivation of several results.

Lemma 1 ([22]). *The following elementary series manupulations holds.*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2 \rfloor} A(k,n-2k), \tag{6}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2 \rfloor} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+2k). \tag{7}$$

Note that this Lemma can be extended in the following way.

Lemma 2 ([10]). *The following elementary series manupulations holds.*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/j \rfloor} A(k,n-jk), \tag{8}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/j \rfloor} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+jk). \tag{9}$$

3. The Fully Degenerate Gould–Hopper Polynomials with a q Parameter

Let $n, j \in \mathbb{Z}$ with $n \geqq 0$ and $j > 0$ and let $q, x, y \in \mathbb{R}/\{0\}$ with $q \neq 0$. We define the fully degenerate Gould–Hopper polynomials with a q parameter by the following generating function to be

$$G(x,y,t) = \sum_{n=0}^{\infty} H_{n,q}^{(j)}(x,y;w)\frac{t^n}{n!} = (1+\omega q t)^{\frac{x}{\omega}}\left(1+\omega q t^j\right)^{\frac{y}{\omega}}. \tag{10}$$

We now examine some special cases of the fully degenerate Gould–Hopper polynomials with a q parameter as follows.

Remark 1.

1. *When $\omega \to 0$, we obtain the Gould–Hopper polynomials with a q parameter denoted by $H_{n,q}^{(j)}(x,y;w)$ (c.f. [10,22,23]).*
2. *When $q \to 1$, we get the fully degenerate Gould–Hopper polynomials denoted by $H_n^{(j)}(x,y;w)$ (see [12,13]).*
3. *When $\omega \to 0$ and $q \to 1$, we have the Gould–Hopper polynomials denoted by $H_n^{(j)}(x,y)$ (c.f. [3,10]).*
4. *Setting $j = 2$ and $q \to 1$, we get the fully degenerate Hermite polynomials denoted by $H_n(x,y;w)$ (c.f. [12,13]).*
5. *When $\omega \to 0$, $j = 2$ and $q \to 1$, we reach the classical Hermite polynomials denoted by $H_n(x,y)$ (see [3,4,10,11,16,20,21,24]).*

Theorem 1. *The fully degenerate Gould–Hopper polynomials with a q parameter have the following representation*

$$H_{n,q}^{(j)}(x,y;w) = n! \sum_{k=0}^{\lfloor n/j \rfloor} \frac{x^{(n-jk,\omega)} y^{(k,\omega)}}{(n-jk)!k!} q^{n-(j-1)k},$$

where $\lfloor \cdot \rfloor$ is the Gauss notation, and represents the maximum integer which does not exceed the number in the square brackets.

Proof. From the generating function of the fully degenerate Gould–Hopper polynomials with a q parameter and the transformation formula (8), we get

$$\begin{aligned} (1+\omega qt)^{\frac{x}{\omega}} \left(1+\omega qt^j\right)^{\frac{y}{\omega}} &= \left(\sum_{n=0}^{\infty} x^{(n,\omega)} \frac{(qt)^n}{n!}\right) \left(\sum_{k=0}^{\infty} y^{(k,\omega)} \frac{q^k t^{kj}}{k!}\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x^{(n,\omega)} y^{(k,\omega)} \frac{(qt)^n}{n!} \frac{q^k t^{kj}}{k!} \\ &= \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{\lfloor n/j \rfloor} \frac{x^{(n-jk,\omega)} y^{(k,\omega)}}{(n-jk)!k!} q^{n-(j-1)k} \right) \frac{t^n}{n!} \end{aligned}$$

□

Theorem 2. *The following difference rules hold true*

$$\Delta_\omega x \, H_{n,q}^{(j)}(x,y;w) = qn \, H_{n-1,q}^{(j)}(x,y;w), \tag{11}$$

$$\Delta_\omega y \, H_{n,q}^{(j)}(x,y;w) = qn^{(j,1)} \, H_{n-j,q}^{(j)}(x,y;w). \tag{12}$$

Proof. It is not difficult to see that $\Delta_\omega x G(x,y,t) = qtG(x,y,t)$. Hence, we get

$$\sum_{n=0}^{\infty} \Delta_\omega x \, H_{n,q}^{(j)}(x,y;w) \frac{t^n}{n!} = \sum_{n=0}^{\infty} q \, H_{n,q}^{(j)}(x,y;w) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} qn \, H_{n-1,q}^{(j)}(x,y;w) \frac{t^n}{n!}.$$

Then (11) is proved. Equation (12) follows in the same way. □

Note that (11) shows that the polynomials $H_{n,\omega,q}^{(j)}(x,y)$ form a Δ_ω-Appell set [8].

Proposition 3. *The following inversion formula holds true.*

$$x^{(n,\omega)} = n! \sum_{k=0}^{\lfloor n/j \rfloor} \frac{q^{(1-j)k} y^{(k,\omega)}}{(n-jk)!k!} H_{n-jk,q}^{(j)}(x,y;\omega).$$

Proof. The proof follows from the equation $\left(1+\omega q^t\right)^{\frac{x}{\omega}} = \left(1+\omega qt^j\right)^{-\frac{y}{\omega}} G(x,y,t)$. □

Proposition 4. *The following addition formula is valid.*

$$H_{n,q}^{(j)}(x_1+x_2, y_1+y_2;\omega) = \sum_{k=0}^{n} \binom{n}{k} H_{k,q}^{(j)}(x_1,y_1;\omega) H_{n-k,q}^{(j)}(x_2,y_2;\omega). \tag{13}$$

Proof. The proof follows from the functional equation $G(x_1+x_2, y_1+y_2, t) = G(x_1,y_1,t)G(x_2,y_2,t)$. □

Proposition 5. *Let a be a non zero complex number, then the following equations is valid*

$$H_{n,q}^{(j)}(ax,ay;\omega)=a^{n}H_{n,q}^{(j)}(x,y;\frac{\omega}{a}).$$

4. The Gould–Hopper Based Fully Degenerate Poly-Bernoulli Polynomials with a q Parameter

Let $n,k,j\in\mathbb{Z}$ with $n\geqq 0$ and $k,j>0$ and let $q,x,y\in\mathbb{R}/\{0\}$ with $q\neq 0$. We introduce the Gould–Hopper-based fully degenerate poly-Bernoulli polynomials with a q parameter by means of the following generating function

$$\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!}=\frac{qLi_{k}\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{1-(1+\omega qt)^{-\frac{1}{\omega}}}\left(1+\omega qt\right)^{\frac{x}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}. \tag{14}$$

Upon setting $x=0=y$, we then get ${}_{H}\beta_{n,q}^{(k,j)}\left(0,0;\omega\right):=\beta_{n,q}^{(k)}\left(\omega\right)$ which are called the fully degenerate poly-Bernoulli numbers with a q parameter, see [13].

Some special cases of ${}_{H}\mathcal{B}_{n,q}^{(k,j)}\left(x,y\right)$ are listed in the following remark.

Remark 2.

1. *When $\omega\to 0$, we obtain the Gould–Hopper-based poly-Bernoulli polynomials with a q parameter denoted by ${}_{H}\beta_{n,q}^{(k,j)}\left(x,y\right)$ (c.f. [10]).*
2. *When $q\to 1$, we get the Gould–Hopper-based fully degenerate poly-Bernoulli polynomials denoted by ${}_{H}\beta_{n}^{(k,j)}\left(x,y;\omega\right)$.*
3. *When $y=0$, we have the fully degenerate poly-Bernoulli polynomials with a q parameter denoted by $\beta_{n,q}^{(k)}\left(x;\omega\right)$ (c.f. [13]).*
4. *When $\omega\to 0$ and $q\to 1$, we reach the Gould–Hopper-based poly-Bernoulli polynomials denoted by ${}_{H}\beta_{n}^{(k,j)}\left(x,y\right)$ (see [10,11,19]).*
5. *When $k=1$, we get the Gould–Hopper-based fully degenerate Bernoulli polynomials with a q parameter denoted by ${}_{H}\beta_{n,q}^{[j]}\left(x,y;\omega\right)$.*
6. *When $\omega\to 0$ and $k=1$, we reach the Gould–Hopper-based Bernoulli polynomials with a q parameter denoted by ${}_{H}\beta_{n,q}^{[j]}\left(x,y\right)$ (see [9,10,20,24]).*
7. *Upon setting $k=1$ and $q\to 1$, we get the Gould–Hopper-based fully degenerate Bernoulli polynomials denoted by ${}_{H}\beta_{n}^{[j]}\left(x,y;\omega\right)$.*
8. *When $k=q\to 1$ and $y=0$, we obtain the fully degenerate Bernoulli polynomials denoted by $\beta_{n}\left(x;\omega\right)$ (see [10,12–15]).*
9. *When $k=q\to 1$, $\omega\to 0$ and $j=2$, we have the Hermite based Bernoulli polynomials denoted by ${}_{H}\beta_{n}\left(x,y\right)$ (c.f. [19,20,24]).*
10. *For $k=q\to 1$, $\omega\to 0$ and $y=0$, we reach the classical Bernoulli polynomials denoted by $B_{n}\left(x\right)$ (see [1,2,25]).*

Proposition 6. *The following connection formula holds true.*

$${}_{H}\beta_{n,q}^{(k,j)}(x,y,\omega)=\sum_{s=0}^{n}\binom{n}{s}\beta_{s,q}^{(k)}(\omega)H_{n-s,q}^{(j)}(x,y;\omega).$$

Proof. The proof follows by applying the Cauchy product. □

Proposition 7. *The following difference rules apply.*

$$\begin{aligned}\Delta_{\omega,x}\left[{}_{H}\beta_{n,q}^{(k,j)}(x,y,\omega)\right] &= qn\,{}_{H}\beta_{n-1,q}^{(k,j)}(x,y,\omega),\\ \Delta_{\omega,y}\left[{}_{H}\beta_{n,q}^{(k,j)}(x,y,\omega)\right] &= qn_{H}^{(j,1)}\beta_{n-j,q}^{(k,j)}(x,y,\omega).\end{aligned}$$

Proposition 8. *The following expansion theorem holds.*

$$ {}_{H}\beta_{n,q}^{(k,j)}(x,y,\omega) = n!\sum_{s=0}^{\lfloor \frac{n}{j} \rfloor}\left(\frac{y}{\omega}\right)^{(s,1)}\frac{\omega^{s}q^{s}}{s!(n-js)!}\,{}_{H}\beta_{n-js,q}^{(k)}(x;\omega). $$

Proof. Indeed,

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{H}\odot_{n,q}^{(k,j)}(x,y,\omega)\frac{t^{n}}{n!} &= \frac{qLi_{k}\left(\frac{1-(1+\omega t)^{-\frac{q}{\omega}}}{q}\right)}{1-(1+\omega t)^{-\frac{q}{\omega}}}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}} \\ &= \left(\sum_{n=0}^{\infty} {}_{H}\beta_{n,q}^{(k)}(x;\omega)\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{y}{\omega}\right)^{(n,1)}q^{n}\omega^{n}\frac{t^{jn}}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\lfloor \frac{n}{j} \rfloor} {}_{H}\beta_{n-js,q}^{(k)}(x;\omega)\frac{t^{n-js}}{(n-js)!}\left(\frac{y}{\omega}\right)^{(s,1)}q^{s}\omega^{s}\frac{t^{js}}{s!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\lfloor \frac{n}{j} \rfloor} {}_{H}\beta_{n-js,q}^{(k)}(x;\omega)\left(\frac{y}{\omega}\right)^{(s,1)}\frac{q^{s}\omega^{s}n!}{s!(n-js)!}\right)\frac{t^{n}}{n!} \end{aligned} $$

which gives the desired result. □

5. The Gould–Hopper Based Fully Degenerate (α, q)-Stirling Numbers of the Second Kind

In this part, we deal with the Gould–Hopper-based fully degenerate (α, q)-Stirling numbers of the second kind and investigate their diverse relations.

Definition 3. *Let $n, m, j \in \mathbb{Z}$ with $n \geqq m \geqq 0$ and $j > 0$ and let $q, \alpha, x, y \in \mathbb{R}/\ \{0\}$ with $q \neq 0$ and $\alpha \neq 0$. The Gould–Hopper based fully degenerate (α, q)-Stirling numbers of the second kind are defined as follows*

$$ \sum_{n=0}^{\infty} S_{2,q}^{(\alpha,j)}(n,m:x,y;\omega)\frac{t^{n}}{n!} = \frac{\left(\alpha\left(1+\omega qt\right)^{\frac{1}{\omega}}-1\right)^{m}}{m!}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}. \tag{15} $$

Remark 3.

1. *When $\omega \to 0$, we obtain the Gould–Hopper-based (α, q)-Stirling numbers of the second kind denoted by $S_{2,q}^{(\alpha,j)}(n,m:x,y)$ (c.f. [10]).*
2. *When $q \to 1$, we get the Gould–Hopper-based fully degenerate α-Stirling numbers of the second kind denoted by $S_{2}^{(\alpha,j)}(n,m:x,y;\omega)$.*
3. *When $y = 0$, we have the fully degenerate (α, q)-Stirling numbers of the second kind denoted by $S_{2,q}^{\alpha}(n,m:x;\omega)$.*
4. *When $\alpha = 1$, we have the Gould–Hopper-based fully degenerate (q)-Stirling numbers of the second kind denoted by $S_{2,q}^{(j)}(n,m:x,y;\omega)$.*
5. *When $\omega \to 0$ and $q \to 1$, we reach the Gould–Hopper-based α-Stirling numbers of the second kind denoted by $S_{2}^{(\alpha)}(n,m:x,y)$[2,10,16]*
6. *When $\omega \to y = 0$, we reach the (α, q)-Stirling numbers of the second kind denoted by $S_{2,q}^{\alpha}(n,m:x)$.*
7. *Upon setting $\omega \to y = 0$ and $q \to 1$, we get the α-Stirling numbers of the second kind denoted by $S_{2}^{\alpha}(n,m:x,y;\omega)$ (c.f. [2,22,25]).*
8. *When $x = y = 0$, we reach the fully degenerate (α, q)-Stirling numbers of the second kind denoted by $S_{2,q}^{(\alpha)}(n,m:w)$.*
9. *For $\omega \to y = x = 0$ and $q \to \alpha = 1$, we reach the familiar Stirling numbers of the second kind denoted by $S_{2}(n,m)$ (see [2,5–7,10,13–16,25]).*

Proposition 9. *The following hold true*

$$\begin{aligned} S_{2,q}^{(\alpha,j)}\left(n,m:x,y;\omega\right) &= \sum_{s=0}^{n}\binom{n}{s}S_{2,q}^{(\alpha)}\left(s,m:w\right)H_{n-s,q}^{(j)}\left(x,y;w\right), \\ S_{2,q}^{(\alpha,j)}\left(n,m:x,y;\omega\right) &= \sum_{s=0}^{n}\binom{n}{s}\omega^{s}q^{s}\left(\frac{x}{\omega}\right)^{(s,1)}S_{2,q}^{(\alpha,j)}\left(n-s,m:0,y;\omega\right), \\ S_{2,q}^{(\alpha,j)}\left(n,m:x,y;\omega\right) &= n!\sum_{s=0}^{\left\lfloor\frac{n}{j}\right\rfloor}\frac{\omega^{s}q^{s}}{s!\left(n-js\right)!}\left(\frac{y}{\omega}\right)^{(s,1)}S_{2,q}^{\alpha}\left(n-js,m:x;\omega\right). \end{aligned}$$

Proposition 10. *The following difference rule are valid*

$$\begin{aligned} \Delta_{\omega,x}\left[S_{s,q}^{(\alpha,j)}(n,m;x,y;\omega)\right] &= qnS_{s,q}^{(\alpha,j)}(n-1,m;x,y;\omega); \\ \Delta_{\omega,y}\left[S_{s,q}^{(\alpha,j)}(n,m;x,y;\omega)\right] &= q(n)^{(j,1)}S_{s,q}^{(\alpha,j)}(n-j,m;x,y;\omega). \end{aligned}$$

6. Some Connection Formulas

In this section, we give multifarious connection formulas including the fully degenerate Gould–Hopper polynomials with a q parameter, the Gould–Hopper-based fully degenerate poly-Bernoulli polynomials with a q parameter and the Gould–Hopper-based fully degenerate (α,q)-Stirling numbers of the second kind.

Theorem 3. *The following connection formula holds*

$$\begin{aligned} &\sum_{s=0}^{n}\binom{n}{s}\omega^{n-s}q^{n-s}\left(\frac{1}{\omega}\right)^{(n-s,1)}{}_{H}\beta_{s,q}^{(k,j)}\left(x,y;\omega\right)-{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right) \\ &\quad=\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^{k}}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{s}H_{n,q}^{(j)}\left(x-s+1,y;\omega\right). \end{aligned}$$

Proof. By (10) and (14), we have

$$\left((1+\omega qt)^{\frac{1}{\omega}}-1\right)\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!}=qLi_{k}\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}. \tag{16}$$

Let LHS and RHS be the left hand-side and the right hand-side of (16), respectively. Then, we get

$$\begin{aligned} LHS &= (1+\omega qt)^{\frac{1}{\omega}}\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!}-\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!} \\ &= \left(\sum_{n=0}^{\infty}\omega^{n}q^{n}\left(\frac{1}{\omega}\right)^{(n,1)}\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!}\right)-\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\omega^{n-s}q^{n-s}\left(\frac{1}{\omega}\right)^{(n-s,1)}{}_{H}\beta_{s,q}^{(k,j)}\left(x,y;\omega\right)\right)\frac{t^{n}}{n!}-\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\frac{t^{n}}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\omega^{n-s}q^{n-s}\left(\frac{1}{\omega}\right)^{(n-s,1)}{}_{H}\beta_{s,q}^{(k,j)}\left(x,y;\omega\right)-{}_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right)\right)\frac{t^{n}}{n!} \end{aligned}$$

and

$$
\begin{aligned}
RHS &= q\sum_{m=1}^{\infty}\frac{\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)^{m}}{m^{k}}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}\\
&= q\sum_{m=0}^{\infty}\frac{\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)^{m+1}}{(m+1)^{k}}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}\\
&= q\sum_{m=0}^{\infty}\frac{q^{-m-1}}{(m+1)^{k}}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{s}(1+\omega qt)^{\frac{x-s+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}\\
&= q\sum_{m=0}^{\infty}\frac{q^{-m-1}}{(m+1)^{k}}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{s}\sum_{n=0}^{\infty}H_{n,q}^{(j)}(x-s+1,y;\omega)\frac{t^{n}}{n!}\\
&= \sum_{n=0}^{\infty}\left(q\sum_{m=0}^{\infty}\frac{q^{-m-1}}{(m+1)^{k}}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{s}H_{n,q}^{(j)}(x-s+1,y;\omega)\right)\frac{t^{n}}{n!}.
\end{aligned}
$$

Combining LHS and RHS gives the asserted result (3). □

We now give the following theorem.

Theorem 4. *We have*

$$
\begin{aligned}
&\sum_{s=0}^{n}\binom{n}{s}\omega^{n-s}q^{n-s}\left(\frac{1}{\omega}\right)^{(n-s,1)}{}_{H}\beta_{s,q}^{(k,j)}(x,y;\omega)-{}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega)\\
&=\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^{k}}q^{-m}(m+1)!S_{2,-q}^{(1,j)}(n,m+1:-x-1,-y;-\omega).
\end{aligned}
$$

Proof. Recall that (16) reads

$$
\left((1+\omega qt)^{\frac{1}{\omega}}-1\right)\sum_{n=0}^{\infty}{}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega)\frac{t^{n}}{n!}=qLi_{k}\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}.
$$

Using (15) and reconsidering the RHS of (16) as

$$
\begin{aligned}
RHS &= q\sum_{m=0}^{\infty}\frac{\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)^{m+1}}{(m+1)^{k}}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{-m}}{(m+1)^{k}}\left((1+\omega qt)^{-\frac{1}{\omega}}-1\right)^{m+1}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{-m}}{(m+1)^{k}}(m+1)!\sum_{n=0}^{\infty}S_{2,-q}^{(j)}(n,m+1:-x-1,-y;-\omega)\frac{t^{n}}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{-m}}{(m+1)^{k}}(m+1)!S_{2,-q}^{(j)}(n,m+1:-x-1,-y;-\omega)\right)\frac{t^{n}}{n!},
\end{aligned}
$$

we obtain the desired result (4). □

We provide the following theorem.

Theorem 5. *We have*

$$
{}_H\beta_{n,q}^{(k,j)}(x,y;\omega) = q\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-q)^{-m-1}(m+1)!}{(m+1)^k}S_{2,-q}^{(j)}(n,m+1:-x+s,-y;-\omega)\,. \tag{17}
$$

Proof. By (14) and (15), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}{}_H\beta_{n,q}^{(k,j)}(x,y;\omega)\frac{t^n}{n!} &= \frac{q}{1-(1+\omega qt)^{-\frac{1}{\omega}}}\sum_{m=0}^{\infty}\frac{q^{-m-1}\left(1-(1+\omega qt)^{-\frac{1}{\omega}}\right)^{m+1}}{(m+1)^k}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}\\
&= \sum_{s=0}^{\infty}(1+\omega qt)^{-\frac{s}{\omega}}\sum_{m=0}^{\infty}\frac{q^{-m}\left(1-(1+\omega qt)^{-\frac{1}{\omega}}\right)^{m+1}}{(m+1)^k}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}(-1)^{m+1}}{(m+1)^k}\left((1+\omega qt)^{-\frac{1}{\omega}}-1\right)^{m+1}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}(-1)^{m+1}}{(m+1)^k}\sum_{n=0}^{\infty}S_{2,-q}^{(j)}(n,m+1:-x+s,-y;-\omega)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}(-1)^{m+1}}{(m+1)^k}S_{2,-q}^{(j)}(n,m+1:-x+s,-y;-\omega)\right)\frac{t^n}{n!},
\end{aligned}
$$

which gives the claimed result (17). □

We have the following theorem.

Theorem 6. *We have*

$$
{}_H\beta_{n,q}^{(k,j)}(x,y;\omega) = \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{u=0}^{m+1}\binom{m+1}{u}(-1)^u H_{n,q}^{(j)}(x-s-u,y;\omega)\,. \tag{18}
$$

Proof. From (10) and (14), we investigate

$$
\begin{aligned}
\sum_{n=0}^{\infty}{}_H\beta_{n,q}^{(k,j)}(x,y;\omega)\frac{t^n}{n!} &= \frac{q}{1-(1+\omega qt)^{-\frac{1}{\omega}}}\sum_{m=0}^{\infty}\frac{q^{-m-1}}{(m+1)^k}\left(1-(1+\omega qt)^{-\frac{1}{\omega}}\right)^{m+1}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{u=0}^{m+1}\binom{m+1}{u}(-1)^u(1+\omega qt)^{\frac{x-s-u}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{u=0}^{m+1}\binom{m+1}{u}(-1)^u\sum_{n=0}^{\infty}H_{n,q}^{(j)}(x-s-u,y;\omega)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{u=0}^{m+1}\binom{m+1}{u}(-1)^u H_{n,q}^{(j)}(x-s-u,y;\omega)\right)\frac{t^n}{n!},
\end{aligned}
$$

which completes the proof of this theorem. □

We state the following theorem.

Theorem 7. *The following relation is valid*

$$
{}_H\beta_{n,q}^{(k,j)}(x,y;\omega) = q\sum_{m=0}^{n}\binom{n}{m}\sum_{l=1}^{m+1}\frac{(-q)^l\,l!}{l^k\,(m+1)}S_{2,-q}(m+1,l:-w)_H\,B_{n-m,q}^{j}(x+1,y;\omega)\,, \tag{19}
$$

where ${}_HB^{j}_{n-m,q}(x,y;\omega)$ *denotes the Gould–Hopper-based degenerate Bernoulli polynomials with a q parameter defined by*

$$\sum_{n=0}^{\infty} {}_HB^{j}_{n,q}(x,y;\omega)\frac{t^n}{n!} = \frac{t}{(1+\omega qt)^{\frac{1}{\omega}}-1}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}.$$

Proof. In view of (14) and (15), we observe

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_H\beta^{(k,j)}_{n,q}(x,y;\omega)\frac{t^n}{n!} &= q\frac{Li_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{t}\frac{t}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}} \\
&= q\left(\frac{Li_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{t}\right)\left(\frac{t}{(1+\omega qt)^{\frac{1}{\omega}}-1}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}\right) \\
&= q\left(\frac{1}{t}\sum_{l=1}^{\infty}\frac{(-q)^l}{l^k}l!\sum_{m=l}^{\infty}S_{2,-q}(m,l:-w)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}{}_HB^{j}_{n,q}(x+1,y;\omega)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\left(q\sum_{l=1}^{m+1}\frac{(-q)^l}{l^k}l!\frac{S_{2,-q}(m+1,l:-w)}{m+1}\,{}_HB^{j}_{n-m,q}(x+1,y;\omega)\right)\frac{t^n}{n!},
\end{aligned}$$

which gives the desired result (19). □

Theorem 8. *We have*

$$ {}_H\beta^{(k,j)}_{n,q}(x+1,y;\omega) - {}_H\beta^{(k,j)}_{n,q}(x,y;\omega) = \sum_{l=0}^{\infty}\frac{q^{-l}(l+1)!}{(l+1)^k}S^{(j)}_{2,q}(n,l+1:x-l-1,y;\omega). \tag{20}$$

Proof. In view of (14), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left({}_H\beta^{(k,j)}_{n,q}(x+1,y;\omega) - {}_H\beta^{(k,j)}_{n,q}(x,y;\omega)\right)\frac{t^n}{n!} &= \frac{qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}} \\
&\quad - \frac{qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}} \\
&= qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)\frac{(1+\omega qt)^{\frac{1}{\omega}}-1}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}} \\
&= qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}} \\
&= \sum_{l=0}^{\infty}\frac{q^{-l}\left(1-(1+\omega qt)^{-\frac{1}{\omega}}\right)^{l+1}}{(l+1)^k}(1+\omega qt)^{\frac{x+1}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}
\end{aligned}$$

$$= \sum_{l=0}^{\infty} \frac{q^{-l}\left(\left(1+\omega qt\right)^{\frac{1}{\omega}}-1\right)^{l+1}}{\left(l+1\right)^{k}} \left(1+\omega qt\right)^{\frac{x-l-1}{\omega}} \left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}$$

$$= \sum_{l=0}^{\infty} \frac{q^{-l}\left(l+1\right)!}{\left(l+1\right)^{k}} \frac{\left(\left(1+\omega qt\right)^{\frac{1}{\omega}}-1\right)^{l+1}}{\left(l+1\right)!} \left(1+\omega qt\right)^{\frac{x-l-1}{\omega}} \left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{l=0}^{\infty} \frac{q^{-l}\left(l+1\right)!}{\left(l+1\right)^{k}} S_{2,q}^{(j)}\left(n, l+1 : x-l-1, y; \omega\right) \right) \frac{t^{n}}{n!},$$

which completes the proof. □

Theorem 9. *We have*

$$_{H}\beta_{n,q}^{(k,j)}\left(x,y;\omega\right) = q \sum_{s=0}^{m-1} \sum_{u=0}^{n} \sum_{l=1}^{u+1} \binom{n}{u} \frac{\left(-1\right)^{l}}{l^{k}} l! \frac{S_{2,-q}\left(u+1, l : -w\right)}{u+1} \, _{H}B_{n-u,mq}^{j}\left(\frac{x+s+1}{m}, \frac{y}{m}; \frac{\omega}{m}\right). \quad (21)$$

Proof. By (14) and (15), we acquire

$$\sum_{n=0}^{\infty} {_{H}\beta_{n,q}^{(k,j)}}\left(x,y;\omega\right) \frac{t^{n}}{n!} = \frac{qLi_{k}\left(\frac{1-\left(1+\omega qt\right)^{-\frac{1}{\omega}}}{q}\right)}{\left(1+\omega qt\right)^{\frac{1}{\omega}}-1} \left(1+\omega qt\right)^{\frac{x+1}{\omega}} \left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}$$

$$= q \frac{Li_{k}\left(\frac{1-\left(1+\omega qt\right)^{-\frac{1}{\omega}}}{q}\right)}{\left(1+\omega qt\right)^{\frac{m}{\omega}}-1} \sum_{s=0}^{m-1} \left(1+\omega qt\right)^{\frac{x+s+1}{\omega}} \left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}$$

$$= q \frac{Li_{k}\left(\frac{1-\left(1+\omega qt\right)^{-\frac{1}{\omega}}}{q}\right)}{t} \sum_{s=0}^{m-1} \frac{t}{\left(1+\omega qt\right)^{\frac{m}{\omega}}-1} \left(1+\omega qt\right)^{\frac{x+s+1}{\omega}} \left(1+\omega qt^{j}\right)^{\frac{y}{\omega}}$$

$$= q \left(\frac{1}{t} \sum_{l=1}^{\infty} \frac{\left(-1\right)^{l}}{l^{k}} l! \sum_{n=s}^{\infty} S_{2,-q}\left(n, l : -w\right) \frac{t^{n}}{n!} \right) \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} {_{H}B_{n,mq}^{j}}\left(\frac{x+s+1}{m}, \frac{y}{m}; \frac{\omega}{m}\right) \frac{t^{n}}{n!}$$

$$= \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} \left(\sum_{u=0}^{n} \binom{n}{u} \left(q \sum_{l=1}^{u+1} \frac{\left(-1\right)^{l}}{l^{k}} l! \frac{S_{2,-q}\left(u+1, l : -w\right)}{u+1} \, _{H}B_{n-u,mq}^{j}\left(\frac{x+s+1}{m}, \frac{y}{m}; \frac{\omega}{m}\right) \right) \right) \frac{t^{n}}{n!},$$

which implies the claimed result (21). □

We now present the following implicit summation formula.

Theorem 10. *We have*

$$_{H}\beta_{n,q}^{(k,j)}\left(x+\lambda, y+\nu; \omega\right) = \sum_{u=0}^{n} \binom{n}{u} \, _{H}\beta_{u,q}^{(k,j)}\left(x,y;\omega\right) H_{n-u,q}^{(j)}\left(\lambda, \nu; w\right). \quad (22)$$

Proof. By (14) and (15), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{H}\beta_{n,q}^{(k,j)}(x+\lambda, y+\nu;\omega)\frac{t^n}{n!} &= \frac{qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x+\lambda}{\omega}}\left(1+\omega qt^j\right)^{\frac{y+\nu}{\omega}} \\
&= \frac{qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}}(1+\omega qt)^{\frac{\lambda}{\omega}}\left(1+\omega qt^j\right)^{\frac{\nu}{\omega}} \\
&= \left(\sum_{n=0}^{\infty} {}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} H_{n,q}^{(j)}(\lambda,\nu;\omega)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\binom{n}{u} {}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega)H_{n-u,q}^{(j)}(\lambda,\nu;w)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients $t^n/n!$ of both sides, we obtain the desired result (22). □

Theorem 11. *The following implicit summation formula holds true:*

$$ {}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega) = \sum_{s=0}^{\left\lfloor \frac{n}{j}\right\rfloor}\sum_{m=0}^{n-js}\binom{n-js}{m} {}_{H}\beta_{n-js-m,q}^{(k,j)}(x,y;\omega)\left(\frac{x}{\omega}\right)^{(m,1)}\left(\frac{y}{\omega}\right)^{(s,1)}\omega^{m+s}q^{m+s}\frac{n!}{s!(n-js)!}. \tag{23}$$

Proof. We derive

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega)\frac{t^n}{n!} = \frac{qLi_k\left(\frac{1-(1+\omega qt)^{-\frac{1}{\omega}}}{q}\right)}{1-(1+\omega qt)^{-\frac{1}{\omega}}}(1+\omega qt)^{\frac{x}{\omega}}\left(1+\omega qt^j\right)^{\frac{y}{\omega}} \\
&= \left(\sum_{n=0}^{\infty} {}_{H}\beta_{n,q}^{(k,j)}(x,y;\omega)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{x}{\omega}\right)^{(n,1)}\omega^n q^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{y}{\omega}\right)^{(n,1)}\omega^n q^n\frac{t^{jn}}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_{H}\beta_{n-m,q}^{(k,j)}(x,y;\omega)\left(\frac{x}{\omega}\right)^{(m,1)}\omega^m q^m\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{y}{\omega}\right)^{(n,1)}\omega^n q^n\frac{t^{jn}}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\left\lfloor \frac{n}{j}\right\rfloor}\sum_{m=0}^{n-js}\binom{n-js}{m} {}_{H}\beta_{n-js-m,q}^{(k,j)}(x,y;\omega)\left(\frac{x}{\omega}\right)^{(m,1)}\left(\frac{y}{\omega}\right)^{(s,1)}\omega^{m+s}q^{m+s}\frac{n!}{s!(n-js)!}\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, the proof of this theorem is completed. □

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