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A collocation method for solving boundary value problems of fractional order

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ABSTRACT

In this work, the Sinc-Collocation Method (SCM) is used to find the approximate solutions of the second-order fractional boundary value problems based on the conformable fractional derivative. For this purpose, a theorem is proved to represent the terms having fractional derivatives in terms of sinc basis functions. To show the effectiveness and accuracy of the method, some special problems are handled and the determined solutions are compared with the approximate solutions arising from using the other numerical methods as well as the exact solutions of the problems.

Keywords: Differential Equations with Fractional Order, Sinc-Collocation Method, Boundary Value Problems, Conformable Fractional Derivative.

1. INTRODUCTION

Fractional calculus is a subject of calculus that involves noninteger order differential and integral operators.

The backround of fractional calculus dates back to the end of the 17th century. In 1695, half-order derivative was mentioned in a letter from L'Hopital to Leibniz [1]. Since then, fractional calculus developed mainly as a pure theoretical field for mathematicians. However, in the last few decades fractional calculus has fastinated the interest of many researchers in several areas [2-9]. Many mathematicians contributed to the development of fractional calculus, therefore many definitions for the fractional derivative are available. The most popular definitions are Riemann-Liouville and Caputo definition of fractional derivatives. Riemann-Liouville and Caputo definitions of α order α^{th} derivative of function f is given as,

$$D_a^{\alpha}\ (f)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\,dx$$

and

$$D_{*,a}^{\alpha}\ (f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}}\,dx$$

respectively, where $\alpha \in [\,n-1, n), n = 1,2, \ldots$

In the last years, Khalil et al.[10] identified a new definition of fractional derivative called the conformable fractional derivative. In [11], Abdeljawad developed the definition of conformable fractional derivative and set basic concepts of this new fractional calculus. For a detailed overview of the conformable fractional derivative and applications, we refer the reader to [12-15] and references there in.

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In particular, in this paper SCM is illustrated to determine the approximate solutions of fractional order boundary value problems in the following form

$$\begin{cases} \mu_2(x)y''(x) + \mu_a(x)y^a + \mu_1(x)y'(x) + \\ \mu_\beta(x)y^\beta(x) + \mu_0(x)y(x) = f(x) \\ y(a) = 0, y(b) = 0 \end{cases} \tag{1}$$

Here $y^{(a)}$ and $y^{(\beta)}$ are the conformable fractional derivative for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$.
Approximate solutions of the equation (1) based on Riemann-Liouville and Caputo derivatives has been studied in several articles with various numerical methods. For example, Variational Iteration Method [16], Adomian Decomposition Method [17], Homotopy Perturbation Method [18], Homotopy Analysis Method [19], Haar Wavelet Method [20] etc. In this paper, we investigate the sinc-collocation method (SCM) to obtain the approximate solution of the equation (1) based on the conformable fractional derivative.

In this paper, SCM is firstly applied to determine the solution of the FBVPs based on conformable fractional derivative. The solution function is expanded to a finite series regarding to the composite translated sinc functions and some unknown coeficients. These unknown coeficients are determined by this method. To show the sufficiency and reliability of the SCM, the method is applied some special FBVPs. Obtained numerical results are compared with the exact ones in addition to ones of other numerical methods. As a result of the comparison one can say that SCM is a strong and hopeful method for finding the approximate solutions of FBVPs.

The paper organized as follows. In section 2, we have illustrated some fundamental definitions and properties of fractional calculus and SCM. In section 3, we use SCM to determine an approximate solution of a general fractional differential equation and obtained results are stated as a new theorems. In section 4, by using tables and graphs some special problems are given to show the abilities of present method. Lastly, in section 5, The paper is ended with a conclusion.

2. PRELIMINARIES

In this section, some fundamental definitions and properties with regard to fractional calculus and sinc basis functions are introduced. For more information, see [21-26].

Definition 1. Let $a \in (n, n+1]$ and f be an $n-$ differentiable function at t, where $t > 0$ Then the conformable fractional derivative of f of order α is defined as

$$T_a(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{([a]-1)}(t + \varepsilon t^{([a]-a)}) - f^{([a]-1)}(t)}{\varepsilon} \tag{2}$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Remark 1. As a consequence of Definition1, one can easily show that

$T_a(f)(t) = t^{([a]-a)} f^{[a]}(t)$

where $a \in (n, n+1]$ and f is $(n+1)$ differentiable at $t > 0$.

Theorem 2. Let $a \in (n, n+1]$ and $f; g$ be $\alpha-$ differentiable at a $t > 0$. Then

1. $T_a(af + bg) = aT_a(f) + bT_a(g)$, for all $a, b \in R$.
2. $T_a(t^p) = pt^{p-a}$, for all $p \in R$.
3. $T_a(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
4. $T_a(fg) = fT_a(g) + gT_a(f)$.
5. $T_a\left(\frac{f}{g}\right) = \frac{gT_a(f) + fT_a(g)}{g^2}$.

Definition 2. The Sinc funtion is defined as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \quad x \in \mathbb{R}$$

Definition 3. The translated sinc function with space knots are given by:

$$S(k,h) = \operatorname{sinc}\left(\frac{x - kh}{h}\right) = \begin{cases} \dfrac{\sin\left(\pi \frac{x - kh}{h}\right)}{\pi \frac{x - kh}{h}}, & x \neq kh \\ 1, & x = kh. \end{cases}$$

where $h > 0$ and $k = 0, \pm 1, \pm 2, \ldots$
For constructing the approximation on (a, b), the conformal map is identified with

$$\phi(z) = \ln\left(\frac{z - a}{b - z}\right)$$

Here, the basis function on (a, b) are determined from

$$S_k(z) = S(k,h)(z)\ o\ \phi(z) = \text{sinc}\left(\frac{\phi(z) - kh}{h}\right)$$

The inverse map of $\omega = \phi(z)$ is

$$z = \phi^{-1}(\omega) = \frac{a + be^{\omega}}{1 + e^{\omega}}$$

the sinc grid points $z_k \in (a,b)$ will be denoted by x_k because they are real. For the evenly spaced knots $\{kh\}_{k=-\infty}^{\infty}$, the image corresponding to these knots is defined by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{\omega kh}}\ , k = 0, \pm 1, \pm 2, \ldots$$

3. THE SINC-COLLOCATION METHOD

Let us consider an approximate solution for $y_n(x)$ in Eq.(1) of the form

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x)\,, n = M + N + 1 \tag{3}$$

Here, $S_k(x)$ is the composite function of $S(k,h)$ and $\phi(x)$. The unknown coefficients c_k in Eq.(3) are obtained with SCM using the following theorems.

Theorem 3. The first two derivatives of $y_n(x)$ are given with

$$\frac{d}{dx} y_n(x) = \sum_{k=-M}^{N} c_k \phi(x) \frac{d}{d\phi} S_k(x) \tag{4}$$

and

$$\frac{d^2}{dx^2} y_n(x) = \sum_{k=-M}^{N} c_k \left(\phi''(x) \frac{d}{d\phi} S_k(x) + \left(\phi'(x)\right)^2 \frac{d^2}{d\phi^2} S_k(x) \right) \tag{5}$$

Theorem 4. The conformable fractional derivatives of order β and α of $y_n(x)$ for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$ are given by

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k x^{1-\beta} \phi'(x) \frac{d}{d\phi} S_k(x) \tag{6}$$

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k x^{2-\alpha} \left(\phi''(x) \frac{d}{d\phi} S_k(x) + \left(\phi'(x)\right)^2 \frac{d^2}{d\phi^2} S_k(x) \right) \tag{7}$$

respectively.

Proof. The conformable fractional derivative of order β of $y_n(x)$ in (3) is written as

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k S_k^{(\beta)}(x).$$

Here, according to Remark 1, we can write

$$S_k^{(\beta)}(x) = x^{1-\beta} S_k'(x).$$

Now, if we use Eq.(4), we obtain

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k x^{1-\beta} \phi'(x) \frac{d}{d\phi} S_k(x)$$

Similarly, we may write the conformable fractional derivative of order α of $y_n(x)$ in(3) as

$$y_n^{(\alpha)} = \sum_{k=-M}^{N} c_k S_k^{(\alpha)}(x).$$

By using Remark 1, we have

$$S_k^{(\alpha)}(x) = x^{2-\alpha} S_k''(x).$$

Then by Eq.(5), we get the desired result

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k x^{2-\alpha} \left(\phi''(x) \frac{d}{d\phi} S_k(x) + \left(\phi'(x)\right)^2 \frac{d^2}{d\phi^2} S_k(x) \right)$$

After relocating each term of Eq.(1) with the approximation illustrated in Eq.(3)-(7) and producting the ending equation by $\{(1/\phi')^2\}$, we determine the system

$$\sum_{k=-M}^{N}\left[c_k\left\{\sum_{i=0}^{2} g_i(x)\ \frac{d^i}{d\phi^i}S_k\right\}\right] = \left(f(x)\left(\frac{1}{\phi'(x)}\right)^2\right)$$

where

$$g_0(x) = \mu_0(x)\left(\frac{1}{\phi'(x)}\right)^2$$

$$g_1(x) = \left[\left(\mu_1(x) + \mu_\beta(x)x^{1-\beta}\right)\left(\frac{1}{\phi'(x)}\right. - (\mu_2(x) + \mu_\alpha(x)x^{2-\alpha})\left(\frac{1}{\phi'(x)}\right)'\right)\right]$$

$$g_2(x) = \mu_2(x) + \mu_\alpha(x)x^{2-\alpha}$$

We know from [25] that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}, \quad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}, \quad \delta_{jk}^{(2)} = \delta_{kj}^{(2)}$$

After taking $x = x_j$, we find the next theorem.

Theorem 5. If the considered approximate solution of BVP (1) is Eq.(3), then the discrete sinc-collocation system for the determination of the unknown coefficients $\{c_k\}_{k=-M}^{N}$

$$\sum_{k=-M}^{N}\left[c_k\left\{\sum_{i=0}^{2}\frac{g_i(x_j)(-1)}{h^i}\ \delta_{jk}^{(i)}\right\}\right] = \left(f(x_i)\left(\frac{1}{\phi'(x_j)}\right)^2\right);\ j = -M,\dots,N \tag{8}$$

Let us define some notations to rewrite the equation system we have obtained in the matrix form.

Let $D(y)$ be a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), \dots, y(x_N)$ and all the other elements are zero and $I^{(i)}$ be the matrices formed by

$$I^{(i)} = \left[\delta_{jk}^{(i)}\right], i = 0, 1, 2$$

where $D, I^{(0)}, I^{(1)}$ and $I^{(2)}$ are $n\,x\,n$ order matrices. For calculating the unknown coefficients c_k in (8), we can write this system using the previous notations in the matrix form

$$Ac = B \tag{9}$$

Here,

$$A = \sum_{i=0}^{2}\frac{1}{h^i}\ D(g_i)I^{(i)}$$

$$B = D\left(\frac{f}{\phi'}\right)1$$

$$c = (c_{-M}, c_{-M+1}, \dots, c_N)^T$$

Now we have a linear system of with n equations given by (9). The unknown coefficients c_k can be determined by solving the system.

4. COMPUTATIONAL EXAMPLES

In this section, we consider three different problems being approximately solved in [20] based on Riemann-Liouville and Caputo fractional differential operator. The exact solutions of those three problems are known and will be investigated by using the present method with Mathematica10 software. In each example, we consider $h = \pi/\sqrt{N},\ N = M$.

Example 1. [20] Let us consider the fractionally damped mechanical oscillator equation in the form of

$$y^{(\alpha)}(x) + \lambda y^{(\beta)}(x) + vy(x) = f(x),\ \ 1 < \alpha \leq 2;\ \ 0 < \beta \leq 1$$

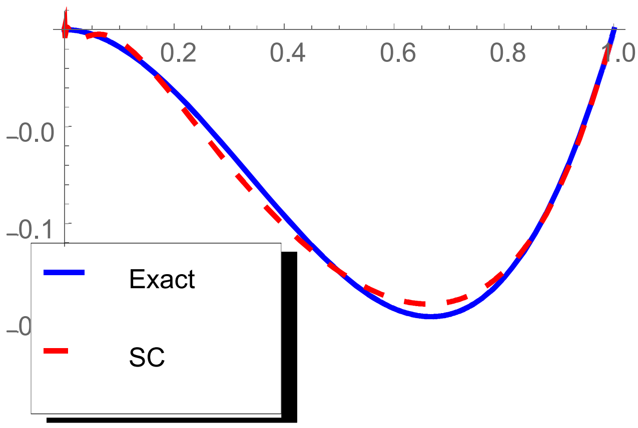
with subject to $y(0) = 0,\ y(1) = 0$. Here, we take $\alpha = \frac{7}{4}, \beta = \frac{1}{2}, \lambda = 1, v = -\frac{1}{\sqrt{\pi}}$ and $f(x) = -\frac{x^3}{\sqrt{\pi}} + 3x^{5/2} + \frac{x^2}{\sqrt{\pi}} - 2x^{3/2} - 6x^{5/4} - 2x^{1/4}$.

$y(x) = x^2(x-1)$ is the exact solution of the problem. The numerical results determined by SCM for this problem are illustrated in Table 1. Also, the comparison of the exact and approximate solutions for various values of N are given graphically in Figure 1.

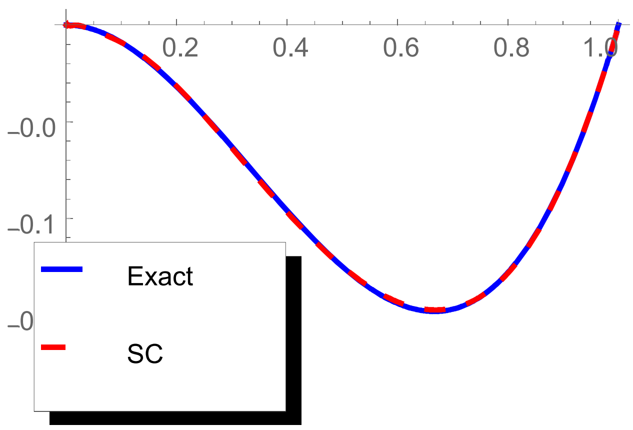
Table 1: Absolute errors for various values of N for Example 1

x	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
0.1	2.996×10^{-5}	6.306×10^{-5}	1.656×10^{-5}	2.624×10^{-8}	3.376×10^{-10}
0.2	3.129×10^{-3}	4.556×10^{-4}	2.219×10^{-5}	3.620×10^{-7}	4.409×10^{-10}

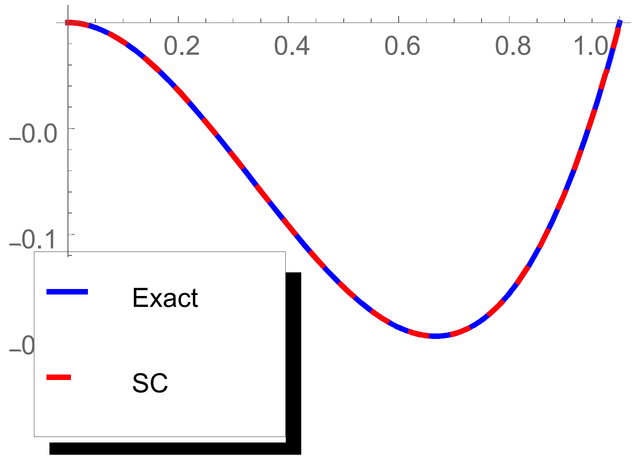
0.3	5.175×10^{-3}	6.260×10^{-4}	8.078×10^{-6}	3.999×10^{-7}	2.295×10^{-10}
0.4	5.196×10^{-3}	8.101×10^{-4}	4.094×10^{-5}	3.222×10^{-7}	4.657×10^{-11}
0.5	1.571×10^{-4}	4.189×10^{-5}	1.689×10^{-6}	1.502×10^{-8}	1.442×10^{-11}
0.6	5.082×10^{-3}	8.232×10^{-4}	4.039×10^{-5}	3.091×10^{-7}	5.855×10^{-11}
0.7	6.423×10^{-3}	5.832×10^{-4}	8.029×10^{-6}	3.721×10^{-7}	2.390×10^{-10}
0.8	3.144×10^{-3}	3.412×10^{-4}	1.944×10^{-5}	3.309×10^{-7}	4.018×10^{-10}
0.9	1.188×10^{-3}	6.109×10^{-5}	1.349×10^{-5}	2.124×10^{-8}	2.862×10^{-10}



(a) $N = 4$



(b) $N = 16$



(c) $N = 64$

Figure 1: Graphs of the exact and the approximate solutions for Example 1

Example 2. [20] Let us assume the fractional differential equation

$$y^{(\alpha)}(x) = y^{(\beta)}(x) - e^{x-1} - 1, 1 < \alpha \leq 2;\ 0 < \beta \leq 1$$

with subject to $y(0) = 0, y(1) = 0$.
$y(x) = x(1 - e^{x-1})$ is the exact solution of the problem for $\alpha = 2$ and $\beta = 1$. In Table 2, the numerical results determined by SCM are compared with the results determined by using Haar wavelet (HWM) and Homotopy perturbation methods (HPM). In addition to presented results in Table 2, the graphs of approximate solutions for various values of α when $\beta = 1$ and $N = 64$ are given in Figure 2. We can easily see that when α approaches to 2, the approximation solutions of fractional order differential equation approach to the solutions of integer order differential equatio via the graphs in Figure 2.

Table 2: Numerical comparisons for Example 2 when $N = 64$, $\alpha = 2$ and $\beta = 1$

x	Fourth order HPM [27]	HWM ($J = 10$)[20]	SCMM	Exact
0.1	0.05934820	0.05934300	0.05934303	0.05934303
0.2	0.11014318	0.11013418	0.11013420	0.11013421
0.3	0.15103441	0.15102438	0.15102440	0.15102441
0.4	0.18048329	0.18047531	0.18047534	0.18047535
0.5	0.19673826	0.19673463	0.19673467	0.19673467
0.6	0.19780653	0.19780792	0.19780797	0.19780797
0.7	0.18142196	0.18142718	0.18142724	0.18142725
0.8	0.14500893	0.14501532	0.14501539	0.14501540
0.9	0.08564186	0.08564623	0.08564632	0.08564632

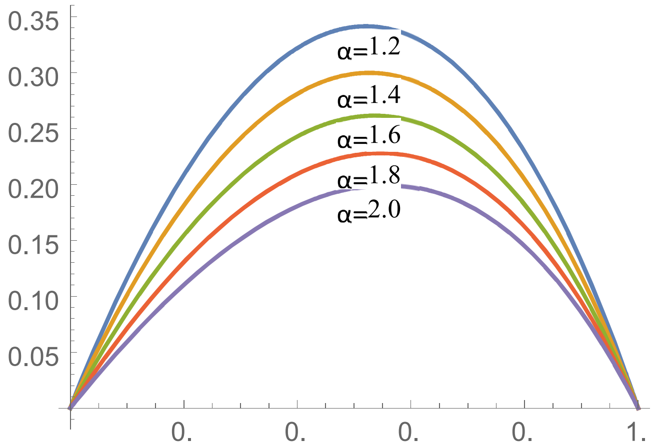


Figure 2: Graphs of the approximate solutions for various values of α for Example 2.

Example 3.[20] Finally, consider the Bagley-Torvik equation the following

$$ay''(x) + by^{(\alpha)}(x) + cy(x) = f(x), 1 < \alpha \leq 2$$

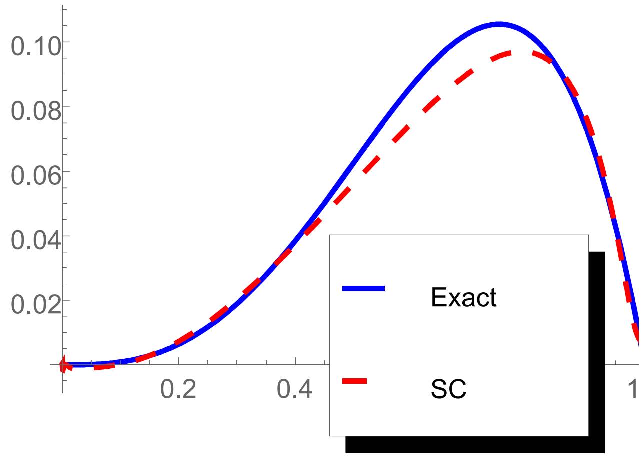
with subject to $y(0) = 0, y(1) = 0$

where $\alpha = \frac{3}{2}, a = 1, b = \frac{8}{17}, c = \frac{13}{51}$ and $f(x) = \frac{1}{51}(360x + 144x^{1.5} - 612x^2 - 288x^{2.5} + 13x^3 - 13x^4)$.

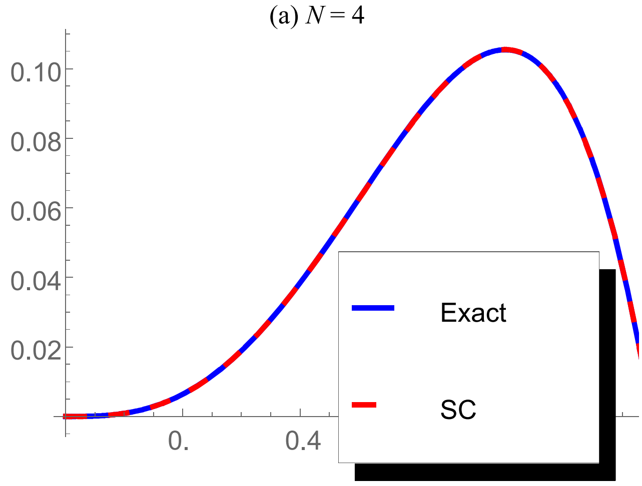
The exact solution of this problem is $y(x) = x^3(1 - x)$. The numerical solutions which are obtained by using SCM for this problem are presented in Table 3. In addition to presented results in Table 3, the graphs of the exact and approximate solutions for various values of N are given in Figure 3.

Table 3: Absolute errors for various values of N for Example 3

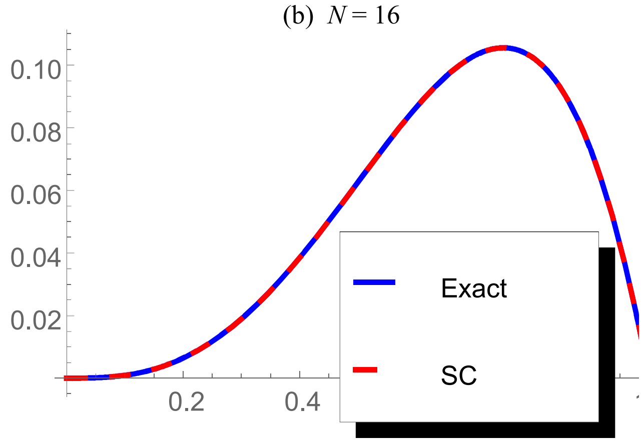
x	N=4	N=8	N=16	N=32	N=64
0.1	6.735×10^{-4}	2.391×10^{-4}	4.792×10^{-6}	8.709×10^{-8}	3.369×10^{-10}
0.2	8.435×10^{-4}	8.087×10^{-5}	1.427×10^{-5}	8.789×10^{-8}	1.422×10^{-10}
0.3	1.406×10^{-3}	3.240×10^{-4}	1.995×10^{-6}	1.728×10^{-7}	1.305×10^{-10}
0.4	1.161×10^{-3}	2.492×10^{-4}	2.353×10^{-5}	2.404×10^{-7}	6.230×10^{-11}
0.5	6.392×10^{-3}	5.979×10^{-4}	2.072×10^{-5}	1.768×10^{-7}	1.856×10^{-10}
0.6	1.148×10^{-2}	1.397×10^{-3}	5.836×10^{-5}	3.949×10^{-7}	1.661×10^{-10}
0.7	1.228×10^{-2}	9.068×10^{-4}	1.759×10^{-5}	6.010×10^{-7}	5.980×10^{-10}
0.8	5.562×10^{-3}	6.948×10^{-4}	2.740×10^{-5}	7.810×10^{-7}	9.854×10^{-10}
0.9	3.363×10^{-3}	2.501×10^{-4}	3.491×10^{-5}	3.870×10^{-8}	9.605×10^{-10}



(a) $N = 4$



(b) $N = 16$



(c) $N = 64$

Figure 3: Graphs of exact and approximate solutions for Example 3

5. CONCLUSION

This study focused on the application of SCM to and the approximate solutions of a class of fractional order two-point boundary value problems. The suggested method is applied to some particular examples to show the applicability and accuracy of the method for FBVPs. Numerical results obtained from the method are compared with the exact solutions and differences are presented in tables and graphical forms. Regarding the results displayed in tables and graphical forms, it can be concluded that SCM is a very effective and convenient method for obtaining the approximate solution of FBVPs.

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