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Research Article

Applications and Properties for Bivariate Bell-Based Frobenius-Type Eulerian Polynomials

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In this study, we introduce sine and cosine Bell-based Frobenius-type Eulerian polynomials, and by presenting several relations and applications, we analyze certain properties. Our first step is to obtain diverse relations and formulas that cover summation formulas, addition formulas, relations with earlier polynomials in the literature, and differentiation rules. Finally, after determining the first few zero values of the Eulerian polynomials, we draw graphical representations of these zero values.

1. Introduction

In recent times, the use of sine and cosine polynomials has led to the definition and construction of generating functions for new families of special polynomials, such as Bernoulli, Euler, and Genocchi; see [1–4]. Fundamental properties and diverse applications for these polynomials have been provided by these types of studies. For instance, not only various implicit and explicit summation formulas, differentiation-integration formulas, symmetric identities, and a lot of relationships with the well-known polynomials have been deeply investigated but also graphical representations of the zero values of these polynomials are drawn after determining them. Moreover, the aforementioned polynomials allow us to investigate worthwhile properties from a very basic procedure and assist to define novel types of special polynomials. Motivated by the above, in this paper, we define the cosine and sine Bell-based Frobenius-type Eulerian polynomials and examine several properties and applications. Our first step is to obtain diverse relations and formulas that cover summation formulas, addition formulas, relations with earlier polynomials in the literature, and differentiation rules. Finally, after determining the first few zero values of the Eulerian polynomials, we draw graphical representations of these zero values.

Let $\xi \in \mathbb{R}$ denotes the set of all real numbers and $\lambda \in \mathbb{C}$ denotes the set of all complex numbers with $\lambda \neq 1$. The Frobenius-type Eulerian polynomial of order $\alpha \in \mathbb{C}$ is introduced as follows (see [5–7]):

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z} = \sum_{j=0}^{\infty} \mathbb{A}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!}, \left|\frac{\log \lambda}{\lambda-1}\right| > |z|. \quad (1)$$

The Frobenius-type Eulerian polynomials have worked by many mathematicians; see [6–11].

Upon setting $\xi = 0$, $\mathbb{A}_j^{(\alpha)}(\lambda) = \mathbb{A}_j^{(\alpha)}(0|\lambda)$ are termed the Frobenius-type Eulerian numbers of order α. In view of (1), it can be readily observed that

$$\mathbb{A}_j^{(\alpha)}(\xi|\lambda) = \sum_{\nu=0}^{j} \binom{j}{\nu} \mathbb{A}_\nu^{(\alpha)}(\lambda)\xi^{j-\nu},$$
$$\mathbb{A}_j^{(\alpha)}(\xi|\lambda) = (\lambda-1)^j \mathbb{H}_j^{(\alpha)}\left(\frac{\xi}{\lambda-1}\middle|\lambda\right), \quad (2)$$

where $\mathbb{H}_j^{(\alpha)}(\xi|\lambda)$ are the Frobenius-Euler polynomials of order α (cf. [11, 12]).

The Stirling numbers of the first kind are introduced for $j \geq 0$ as follows (cf. [13–15]):

$$(\xi)_j = \sum_{p=0}^{j} S_1(j, p)\xi^p, \tag{3}$$

where $(\xi)_0 := 1$ and $(\xi)_j := (\xi - j + 1)(\xi - j + 2) \cdots (\xi - 1)\xi$, $(j \geq 1)$. By (3), we acquire that (see [14, 16, 17])

$$\frac{1}{r!}(\log (1+z))^r = \sum_{j=r}^{\infty} S_1(j, r)\frac{z^j}{j!}, (r \geq 0). \tag{4}$$

The Stirling numbers of the second kind are given for $j \geq 0$ as follows (see [5, 18]):

$$\xi^j = \sum_{q=0}^{j} S_2(j, q)(\xi)_q. \tag{5}$$

In terms of (5), it is easily shown that

$$\sum_{j=r}^{\infty} S_2(j, r)\frac{z^j}{j!} = \frac{(e^z - 1)^r}{r!}. \tag{6}$$

For any nonnegative integer q, the q-Stirling numbers $S_q(j, k)$ of the second kind are introduced as follows (see [19]):

$$\frac{1}{k!}e^{qz}(e^z - 1)^k = \sum_{j=k}^{\infty} S_2(j+q, k+q)\frac{z^j}{j!}. \tag{7}$$

Let q be any nonnegative integer. The Bell-based Stirling polynomials of the second kind are provided as follows (see [13]):

$$\frac{1}{k!}e^{q(e^z-1)}(e^z - 1)^k = \sum_{j=k}^{\infty} {}_{Bel}S_2(j, k : q)\frac{z^j}{j!}. \tag{8}$$

The Apostol types of the Bernoulli $\mathbb{B}_j^{(\alpha)}(\xi|\lambda)$, the Euler $\mathbb{E}_j^{(\alpha)}(\xi|\lambda)$, and the Genocchi polynomials $\mathbb{G}_j^{(\alpha)}(\xi|\lambda)$ of order α are introduced as follows (cf. [11, 17, 20]):

$$\begin{aligned}
\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{\xi z} &= \sum_{j=0}^{\infty} \mathbb{B}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!}(|z + \log \lambda| < 2\pi), \\
\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{\xi z} &= \sum_{j=0}^{\infty} \mathbb{E}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!}(|z + \log \lambda| < \pi), \\
\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} e^{\xi z} &= \sum_{j=0}^{\infty} \mathbb{G}_j^{(\alpha)}(\xi|\lambda)\frac{z^j}{j!}, (|z + \log \lambda| < \pi).
\end{aligned} \tag{9}$$

Also, their corresponding numbers are determined by

$$\mathbb{B}_j^{(\alpha)}(0|\lambda) := \mathbb{B}_j^{(\alpha)}(\lambda), \mathbb{E}_j^{(\alpha)}(0|\lambda) := \mathbb{E}_j^{(\alpha)}(\lambda), \mathbb{G}_j^{(\alpha)}(0|\lambda) := \mathbb{G}_j^{(\alpha)}(\lambda), \tag{10}$$

respectively. In addition, their familiar polynomials and numbers are determined by just choosing $\lambda = \alpha = 1$ in their definitions and shown by $B_j(\xi)$ and $E_j(\xi)$.

The Bell polynomials $\mathrm{Bel}_j(\xi)$ are introduced as follows (see [18, 21–25]):

$$e^{\xi(e^z-1)} = \sum_{j=0}^{\infty} \mathrm{Bel}_j(\xi)\frac{z^j}{j!}. \tag{11}$$

Also, the corresponding Bell numbers are determined by $\mathrm{Bel}_j(1) := \mathrm{Bel}_j$, $(j \geq 0)$. In terms of (6) and (11), it is seen that

$$\mathrm{Bel}_j(\xi) = \sum_{k=0}^{j} S_2(j, k)\xi^k (j \geq 0). \tag{12}$$

In recent years, Duran et al. [13] considered the Bell-based Bernoulli polynomials of order α ${}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi ; \eta)$ given by

$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi ; \eta)\frac{z^j}{j!} = e^{\xi z + \eta(e^z-1)}\left(\frac{z}{e^z - 1}\right)^{\alpha}, \tag{13}$$

which also provides that

$${}_{Bel}\mathbb{B}_j^{(\alpha)}(\xi ; \eta) = \sum_{r=0}^{j} \binom{j}{r} \mathbb{B}_{j-r}^{(\alpha)}(\xi)\mathrm{Bel}_r(\eta). \tag{14}$$

Also, in [13], the authors proved several properties and relations for the aforesaid polynomials. In addition, they gave many quirky formulas arising from the theory of umbral calculus.

Kim et al. [3] and Jamei et al. [1] considered the Bernoulli polynomials and the Euler polynomials based on the cosine and sine polynomials as follows:

$$\frac{1}{2}\sum_{j=0}^{\infty}(\mathbb{B}_j(\xi + i\eta) + \mathbb{B}_j(\xi - i\eta))\frac{z^j}{j!} = \sum_{j=0}^{\infty} \mathbb{B}_j^{(c)}(\xi, \eta)\frac{z^j}{j!} = e^{\xi z} \cos \eta z \frac{z}{e^z - 1}, \tag{15}$$

$$\frac{1}{2i}\sum_{j=0}^{\infty}(\mathbb{B}_j(\xi + i\eta) - \mathbb{B}_j(\xi - i\eta))\frac{z^j}{j!} = \sum_{j=0}^{\infty} \mathbb{B}_j^{(s)}(\xi, \eta)\frac{z^j}{j!} = e^{\xi z} \sin \eta z \frac{z}{e^z - 1}, \tag{16}$$

$$\frac{1}{2}\sum_{j=0}^{\infty}(\mathbb{E}_j(\xi + i\eta) + \mathbb{E}_j(\xi - i\eta))\frac{z^j}{j!} = \sum_{j=0}^{\infty} \mathbb{E}_j^{(c)}(\xi, \eta)\frac{z^j}{j!} = e^{\xi z} \cos \eta z \frac{2}{e^z + 1}, \tag{17}$$

$$\frac{1}{2i}\sum_{j=0}^{\infty}\left(\mathbb{E}_j(\xi+i\eta)-\mathbb{E}_j(\xi-i\eta)\right)\frac{z^j}{j!}=\sum_{j=0}^{\infty}\mathbb{E}_j^{(s)}(\xi,\eta)\frac{z^j}{j!}=e^{\xi z}\sin\eta z\frac{2}{e^z+1}, \tag{18}$$

respectively. In addition, they investigated many relations for the polynomials given above.

The trigonometric polynomials, cosine, and sine polynomials are introduced as follows (see [2–4, 7]):

$$\begin{aligned} e^{\xi z}\cos\eta z&=\sum_{r=0}^{\infty}C_r(\xi,\eta)\frac{z^r}{r!},\\ e^{\xi z}\sin\eta z&=\sum_{r=0}^{\infty}S_r(\xi,\eta)\frac{z^r}{r!}, \end{aligned} \tag{19}$$

which satisfy the following expansion formulas:

$$\begin{aligned} C_r(\xi,\eta)&=\sum_{j=0}^{\lfloor r/2\rfloor}(-1)^j\binom{r}{2j}\xi^{r-2j}\eta^{2j},\\ S_r(\xi,\eta)&=\sum_{j=0}^{\lfloor (r-1)/2\rfloor}\binom{r}{2j+1}(-1)^j\xi^{r-2j-1}\eta^{2j+1}, \end{aligned} \tag{20}$$

where the value of $\lfloor y\rfloor$ is the largest integer that is equal or less than y.

2. Cosine and Sine Bell-Based Frobenius-Type Eulerian Polynomials

Here, we introduce the cosine and sine Bell-based Frobenius-type Eulerian numbers and polynomials, and then we derive several properties and identities for the above polynomials.

Motivated and inspired by the definitions (13) and (15)–(18), we first consider the Bell-based Frobenius-type Eulerian polynomials defined as follows:

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{z\xi}e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi,\zeta|\lambda)\frac{z^j}{j!}. \tag{21}$$

By (21) and the following well-known formula

$$e^{i\eta z}=(\cos\eta z+i\sin\eta z), \tag{22}$$

thus, we have

$$\begin{aligned} \sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta|\lambda)\frac{z^j}{j!}&=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{(\xi+i\eta)z}e^{\zeta(e^z-1)}\\ &=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\zeta(e^z-1)e^{\xi z}(\cos\eta z+i\sin\eta z)}, \end{aligned} \tag{23}$$

$$\begin{aligned} \sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta|\lambda)\frac{z^j}{j!}&=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{(\xi-i\eta)z}e^{\zeta(e^z-1)}\\ &=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}(\cos\eta z-i\sin\eta z)e^{\xi z}e^{\zeta(e^z-1)}. \end{aligned} \tag{24}$$

From (23) and (24), we get

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}\left(\frac{{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta\mid\lambda)+{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta\mid\lambda)}{2}\right)\frac{z^j}{j!}, \tag{25}$$

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}=\sum_{j=0}^{\infty}\left(\frac{{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta\mid\lambda)-{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta\mid\lambda)}{2i}\right)\frac{z^j}{j!}. \tag{26}$$

Hence, here is our definition.

Definition 1. We consider the cosine and sine Bell-based Frobenius-type Eulerian polynomials of order α, for nonnegative integer j, as follows:

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\cos\eta z e^{\zeta(e^z-1)}, \tag{27}$$

$$\sum_{j=0}^{\infty}{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}=\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{\xi z}\sin\eta z e^{\zeta(e^z-1)}, \tag{28}$$

respectively.

Note that ${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,0,0|\lambda)\coloneqq\mathbb{A}_j^{(\alpha)}(\xi|\lambda)$ and ${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,0,0|\lambda)=0(j\geq 0)$.

From (25)–(28), we have

$$\begin{aligned} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)&=\frac{1}{2}\left({}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta|\lambda)+{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta|\lambda)\right),\\ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)&=\frac{1}{2i}\left({}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi+i\eta,\zeta|\lambda)-{}_{Bel}\mathbb{A}_j^{(\alpha)}(\xi-i\eta,\zeta|\lambda)\right). \end{aligned} \tag{29}$$

Remark 2. For $\zeta=\xi=0$ in (27) and (28), we get novel type of the polynomials $\mathbb{A}_j^{(\alpha,c)}(\eta|\lambda)$ and $\mathbb{A}_j^{(\alpha,s)}(\eta|\lambda)$ as

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\cos\eta z=\sum_{j=0}^{\infty}\mathbb{A}_j^{(\alpha,c)}(\eta|\lambda)\frac{z^j}{j!}, \tag{30}$$

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\sin\eta z=\sum_{j=0}^{\infty}\mathbb{A}_j^{(\alpha,s)}(\eta|\lambda)\frac{z^j}{j!}. \tag{31}$$

It is readily observed that (for $j\geq 0$)

$$\mathbb{A}_j^{(\alpha,c)}(0|\lambda)=\mathbb{A}_j^{(\alpha,c)}(\lambda)\text{ and }\mathbb{A}_j^{(\alpha,s)}(0|\lambda)=0. \tag{32}$$

Remark 3. Putting $\zeta = 0$ in (27) and (28), we attain cosine $\mathbb{A}_j^{(\alpha,c)}(\xi,\eta|\lambda)$ and sine $\mathbb{A}_j^{(\alpha,s)}(\xi,\eta|\lambda)$ Frobenius-type Eulerian polynomials:

$$\sum_{j=0}^{\infty} \mathbb{A}_j^{(\alpha,c)}(\xi,\eta|\lambda)\frac{z^j}{j!} = \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z}\cos \eta z,$$
$$\sum_{j=0}^{\infty} \mathbb{A}_j^{(\alpha,s)}(\xi,\eta|\lambda)\frac{z^j}{j!} = \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z}\sin \eta z, \tag{33}$$

respectively.

Remark 4. Letting $\xi = 0$ in (27) and (28), we have novel kind cosine and sine Bell-based Frobenius-type Eulerian polynomials as

$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\eta,\zeta|\lambda)\frac{z^j}{j!} = \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} \cos \eta z e^{\zeta(e^z-1)},$$
$$\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\eta,\zeta|\lambda)\frac{z^j}{j!} = \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} \sin \eta z e^{\zeta(e^z-1)}, \tag{34}$$

respectively.

Remark 5. On setting $\xi = \eta = 0$ in (27) and (28), we attain the Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(\alpha)}(\zeta|\lambda)$ as

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\zeta(e^z-1)} = \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha)}(\zeta|\lambda)\frac{z^j}{j!}. \tag{35}$$

Theorem 6. *Let $j \geq 0$. We acquire that*

$${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{v=0}^{\left[\frac{j}{2}\right]} \binom{j}{2v} (-1)^v \eta^{2v}\, {}_{Bel}\mathbb{A}_{j-2v}^{(\alpha)}(\xi,\zeta|\lambda), \tag{36}$$

$${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{v=0}^{\left[\frac{j-1}{2}\right]} \binom{j}{2v+1} (-1)^v \eta^{2v+1}\, {}_{Bel}\mathbb{A}_{j-2v-1}^{(\alpha)}(\xi,\zeta|\lambda). \tag{37}$$

Proof. It is seen from (30) and (31) that

$$\begin{aligned}\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z}\cos \eta z e^{\zeta(e^z-1)} \\ &= \sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha)}(\xi,\zeta|\lambda)\frac{z^j}{j!}\sum_{v=0}^{\infty}(-1)^v\eta^{2v}\frac{z^v}{(2v)!} \\ &= \sum_{j=0}^{\infty}\left(\sum_{v=0}^{\left[\frac{j}{2}\right]}(-1)^v\eta^{2v}\binom{j}{2v}{}_{\mathrm{Bel}}\mathbb{A}_{j-2v}^{(\alpha)}(\xi,\zeta|\lambda)\right)\frac{z^j}{j!},\end{aligned} \tag{38}$$

$$\begin{aligned}\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z}\sin \eta z e^{\zeta(e^z-1)} \\ &= \sum_{j=0}^{\infty}\left(\sum_{v=0}^{\left[\frac{j-1}{2}\right]}\binom{j}{2v+1}(-1)^v\eta^{2v+1}{}_{Bel}\mathbb{A}_{j-2v-1}^{(\alpha)}(\xi,\zeta|\lambda)\right)\frac{z^j}{j!}.\end{aligned} \tag{39}$$

We acquire the asserted results (36) and (37) in accordance with (38) and (39). □

Theorem 7. *Let $j \geq 0$. We acquire that*

$${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta+x|\lambda) = \sum_{k=0}^{j} {}_{Bel}\mathbb{A}_k^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)Bel_{j-k}(x), \tag{40}$$

$${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta+x|\lambda) = \sum_{k=0}^{j} {}_{Bel}\mathbb{A}_k^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)Bel_{j-k}(x). \tag{41}$$

Proof. By using (11), (23), and (24), we can readily derive (40) and (41) by utilizing series methods. Therefore, we exclude the proofs. □

Theorem 8. *For $j \geq 0$, we have*

$${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j} \binom{j}{u} {}_{Bel}\mathbb{A}_{j-u}^{(\alpha)}(\zeta|\lambda)C_u(\xi,\eta), \tag{42}$$

$${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j} \binom{j}{u} {}_{Bel}\mathbb{A}_{j-u}^{(\alpha)}(\zeta|\lambda)S_u(\xi,\eta). \tag{43}$$

Proof. Utilizing the Cauchy product rule

$$\left(\sum_{j=0}^{\infty} a_j\frac{z^j}{j!}\right)\left(\sum_{v=0}^{\infty} b_v\frac{z^v}{v!}\right) = \sum_{j=0}^{\infty}\left(\sum_{v=0}^{j}\binom{j}{v}a_{j-v}b_v\right)\frac{z^j}{j!}, \tag{44}$$

we investigate

$$\begin{aligned}\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= e^{\xi z}\cos \eta z e^{\zeta(e^z-1)}\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} \\ &= \left(\sum_{v=0}^{\infty} C_v(\xi,\eta)\frac{z^v}{v!}\right)\left(\sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha)}(\zeta|\lambda)\frac{z^j}{j!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{h=0}^{j}\binom{j}{h}{}_{\mathrm{Bel}}\mathbb{A}_{j-h}^{(\alpha)}(\zeta|\lambda)C_h(\xi,\eta)\right)\frac{z^j}{j!},\end{aligned} \tag{45}$$

which implies (42). The other proof (43) can be done similarly. □

Theorem 9. *For $j \geq 0$, we attain*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} Bel_{j-k}(\zeta)\mathbb{A}_k^{(\alpha,c)}(\xi,\eta|\lambda), \quad (46) $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} Bel_{j-k}(\zeta)\mathbb{A}_k^{(\alpha,s)}(\xi,\eta|\lambda), \quad (47) $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} \xi^{j-k}\,{}_{Bel}\mathbb{A}_k^{(\alpha,c)}(\eta,\zeta|\lambda), \quad (48) $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{k=0}^{j} \xi^{j-k}\,{}_{Bel}\mathbb{A}_k^{(\alpha,s)}(\eta,\zeta|\lambda). \quad (49) $$

Proof. Using (27) and (28), the proofs of (46)–(49) can be shown similarly to the proofs of the above theorems. Therefore, we exclude the proof. □

Theorem 10. *For $j \geq 0$, we have*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(s+\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j} \binom{j}{u}\,{}_{Bel}\mathbb{A}_u^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)s^{j-u}, \quad (50) $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi+s,\eta,\zeta|\lambda) = \sum_{u=0}^{j} \binom{j}{u}\,{}_{Bel}\mathbb{A}_u^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)s^{j-u}. \quad (51) $$

Proof. By (27), we attain

$$ \begin{aligned} \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi+s,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{\xi z}e^{\zeta(e^z-1)}\cos \eta z e^{sz} \\ &= \left(\sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}\right)\left(\sum_{u=0}^{\infty} s^u\frac{z^u}{u!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{u=0}^{j}\binom{j}{u}\,{}_{\mathrm{Bel}}\mathbb{A}_u^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)s^{j-u}\right)\frac{z^j}{j!}, \end{aligned} \quad (52) $$

which completes the proof (50). The result (51) can be done similarly. □

Theorem 11. *For $j \geq 0$, we have*

$$ \frac{\partial}{\partial \xi}\,{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = j\,{}_{Bel}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda), \quad (53) $$

$$ \frac{\partial}{\partial \eta}\,{}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = -j\,{}_{Bel}\mathbb{A}_{j-1}^{(\alpha,s)}(\xi,\eta,\zeta|\lambda), \quad (54) $$

$$ \frac{\partial}{\partial \xi}\,{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = j\,{}_{Bel}\mathbb{A}_{j-1}^{(\alpha,s)}(\xi,\eta,\zeta|\lambda), \quad (55) $$

$$ \frac{\partial}{\partial \eta}\,{}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = j\,{}_{Bel}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda). \quad (56) $$

Proof. By means of (27), we compute that

$$ \begin{aligned} \sum_{j=1}^{\infty} \frac{\partial}{\partial \xi}\,{}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} z e^{\xi z}\cos \eta z e^{\zeta(e^z-1)} \\ &= \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^{j+1}}{j!} \\ &= \sum_{j=1}^{\infty} {}_{Bel}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{(j-1)!} \\ &= \sum_{j=1}^{\infty} j\,{}_{\mathrm{Bel}}\mathbb{A}_{j-1}^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!}, \end{aligned} \quad (57) $$

which means (53). The formulas (54), (55), and (56) can be derived similarly. □

Theorem 12. *For $j \geq 0$, we attain*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j}\sum_{k=0}^{u} \binom{j}{u}\,{}_{Bel}\mathbb{A}_{j-u}^{(\alpha,c)}(\eta,\zeta|\lambda)(\xi)_k S_2(u,k), \quad (58) $$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j}\sum_{k=0}^{u} \binom{j}{u}\,{}_{Bel}\mathbb{A}_{j-u}^{(\alpha,s)}(\eta,\zeta|\lambda)(\xi)_k S_2(u,k). \quad (59) $$

Proof. Using (6) and (27), we find

$$ \begin{aligned} \sum_{j=0}^{\infty} {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}(e^z-1+1)^{\xi}\cos \eta z e^{\zeta(e^z-1)} \\ &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\cos \eta z e^{\zeta(e^z-1)}\sum_{k=0}^{\infty}(\xi)_k\frac{(e^z-1)^k}{k!} \\ &= \left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}\cos \eta z e^{\zeta(e^z-1)}\sum_{k=0}^{\infty}(\xi)_k\sum_{u=k}^{\infty}S_2(u,k)\frac{z^u}{u!} \\ &= \sum_{j=0}^{\infty} {}_{\mathrm{Bel}}\mathbb{A}_j^{(\alpha,c)}(\eta;\zeta|\lambda)\frac{z^j}{j!}\sum_{u=0}^{\infty}\sum_{k=0}^{u}(\xi)_k S_2(u,k)\frac{z^u}{u!} \\ &= \sum_{j=0}^{\infty}\sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u}\,{}_{\mathrm{Bel}}\mathbb{A}_{j-u}^{(\alpha,c)}(\eta,\zeta|\lambda)(\xi)_k S_2(u,k)\frac{z^j}{j!}. \end{aligned} \quad (60) $$

In view of (27) and (60), we attain the claimed result (58). Also, we can easily obtain (59) in a similar way. □

We give a relation with the Bell-Stirling polynomials of the second kind as follows.

Table 1: Real and complex zeros of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5\,; 2)$.

Degree	Real zeros	Complex zeros
1	-9	—
2	-13.79583, -4.20417	—
3	-17.50929, -8.57862, -0.91208	—
4	-21.37817, 2.60420	$-8.61301 + 2.46740\,i$, $-8.61301 - 2.46740\,i$
5	-25.2312, -10.1799, 6.23115	$-7.91002 - 4.42772\,i$, $-7.91002 + 4.42772\,i$
6	-29.0774, 9.91421	$-10.306 - 1.41814\,i$, $-10.306 + 1.41814\,i$, $-7.11246 - 6.21942\,i$, $-7.11246 + 6.21942\,i$
7	-32.918, -12.0177, 13.6309	$-9.56347 - 3.17179\,i$, $-9.56347 + 3.17179\,i$, $-6.2841 - 7.82954\,i$, $-6.2841 + 7.82954\,i$
8	-36.7547, -12.8049, -11.0126, 17.3697	$-8.97353 - 4.95183\,i$, $-8.97353 + 4.95183\,i$, $-5.42516 - 9.29329\,i$, $-5.42516 + 9.29329\,i$,
9	-40.5884, -14.3645, 21.1239	$-10.6868 - 2.38861\,i$, $-10.6868 + 2.38861\,i$, $-8.35989 - 6.57959\,i$, $-8.35989 + 6.57959\,i$, $-4.53883 - 10.6389\,i$, $-4.53883 + 10.6389\,i$
10	-44.4198, -15.6243, -11.5504, 4.8894	$-10.3149 - 4.14379\,i$, $-10.3149 + 4.14379\,i$, $-7.70471 - 8.09496\,i$, $-7.70471 + 8.09496\,i$, $-3.62785 - 11.8863\,i$, $-3.62785 + 11.8863\,i$
11	-48.24939, -16.94854, 28.66337	$-11.660956 - 1.92837\,i$, $-11.660956 + 1.92837\,i$, $-9.86402 - 5.74486\,i$, $-9.86402 + 5.74486\,i$, $-7.01359 - 9.51912\,i$, $-7.01359 + 9.51912\,i$, $-2.6941 - 13.0504\,i$, $-2.6941 + 13.0504\,i$
12	-52.0776, -18.24958, -12.42031, 32.444	$-11.46025 - 3.57703\,i$, $-11.46025 + 3.57703\,i$, $5694 - 7.25491\,i$, $-9.35694 + 7.25491\,i$, $-6.292007 - 10.86579\,i$, $-6.292007 + 10.86579\,i$, $-1.73903 - 14.14248\,i$, $-1.73903 + 14.14248\,i$
13	-55.90467, -19.5518, 36.22987	$-12.64336 - 1.62677\,i$, $-12.64336 + 1.62677\,i$, $-11.12894 - 5.13136\,i$, $-11.12894 + 5.13136\,i$, $-8.80659 - 8.69073\,i$, $-8.80659 + 8.69073\,i$, $-5.54426 - 12.14505\,i$, $-5.54426 + 12.14505\,i$, $-0.76349 - 15.17179\,i$, $-0.76349 + 15.17179\,i$
14	-59.7308, -20.84998, -13.40199, 40.0199	$-12.52359 - 3.14914\,i$, $-12.52359 + 3.14914\,i$, $-10.73302 - 6.62043\,i$, $-10.73302 + 6.62043\,i$, $-8.2199 - 10.0621\,i$, $-8.2199 - 10.0621\,i$, $-4.77378 - 13.36466\,i$, $-4.77378 + 13.36466\,i$, $0.23173 - 16.145982\,i$, $0.23173 + 16.145982\,i$
15	-63.55622, -22.14583, 43.81356 $-3.98323 - 14.53071\,i$, $-3.98323 + 14.53071\,i$,	$-13.65616 - 1.38254\,i$, $-13.65616 + 1.38254\,i$, $-12.27016 - 4.65428\,i$, $-12.27016 + 4.65428\,i$, $-10.29033 - 8.04947\,i$, $-10.29033 + 8.04947\,i$, $-7.60183 - 11.37668\,i$, $-7.60183 + 11.37668\,i$ $1.24599 - 17.07154\,i$, $1.24599 + 17.07154\,i$
16	-67.381, -23.4394, -14.4645, 47.61	$-13.5438 - 2.79653\,i$, $-13.5438 + 2.79653\,i$, $-11.9582 - 6.11878\,i$, $-11.9582 + 6.11878\,i$, $-9.80817 - 9.42427\,i$, $-9.80817 + 9.42427\,i$, $-6.95627 - 12.6403\,i$, $-6.95627 + 12.6403\,i$, $-3.17465 - 15.6481\,i$, $-3.17465 + 15.6481\,i$, $2.27862 - 17.9541\,i$, $2.27862 + 17.9541\,i$

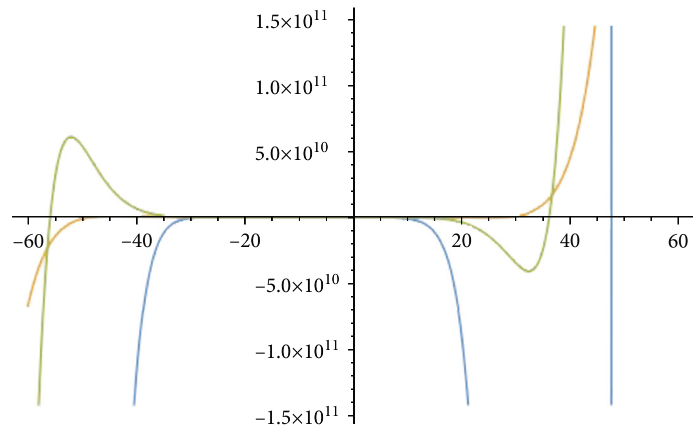


FIGURE 1: Parametric cosine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5; 2)$ for $j = 11$ (orange),13 (green), and16 (blue).

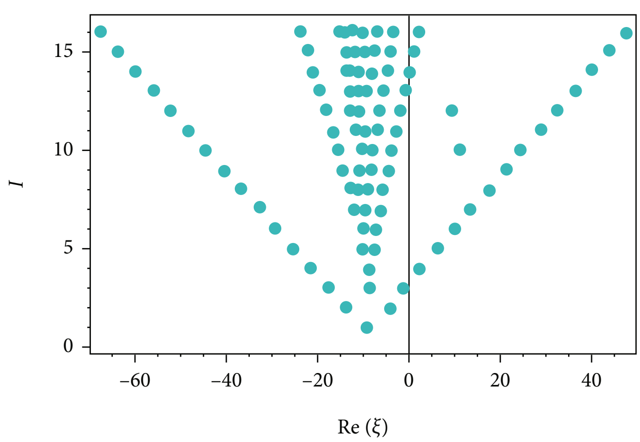


FIGURE 2: Structure of real zeros of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5; 2)$, $j \leq 1 \leq 16$.

Theorem 13. *For $j \geq 0$, we attain*

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u}\mathbb{A}_{j-u}^{(\alpha,c)}(\eta|\lambda)\,{}_{Bel}S_2(u,k:\zeta)(\xi)_k, \tag{61}$$

$$ {}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \sum_{u=0}^{j}\sum_{k=0}^{u}\binom{j}{u}\mathbb{A}_{j-u}^{(\alpha,s)}(\eta|\lambda)\,{}_{Bel}S_2(u,k:\zeta)(\xi)_k. \tag{62}$$

Proof. Using (8), (27), and (28), the proofs of (61) and (62) can be shown similar to the proofs of Theorem 12. So, we omit the proofs. □

3. Some Values with Graphical Representations and Zeros of Sine and Cosine Bell-Based Frobenius-Type Eulerian Polynomials

Here, we indicate the first few sine and cosine Bell-based Frobenius-type Eulerian polynomials with beautiful graphical representations and examine some zero values of these polynomials ${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)$ and ${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)$.

It is not difficult to check that the first five parametric kinds of ${}_{Bel}\mathbb{A}_j^{(\alpha,c)}(\xi,\eta,\zeta|\lambda)$ are

$$ {}_{Bel}\mathbb{A}_0^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = 1,$$

$$ {}_{Bel}\mathbb{A}_1^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = \xi + \zeta + \alpha,$$

$$\begin{aligned} {}_{Bel}\mathbb{A}_2^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = {} & \frac{\zeta}{2} + \frac{\zeta^2}{2} + \alpha\zeta + \zeta\xi + \frac{1}{2}\xi^2 \\ & + \alpha\xi + \frac{1}{2}\lambda\alpha + \frac{1}{2}\alpha^2 - \frac{1}{2}\eta^2, \end{aligned}$$

$$\begin{aligned} {}_{Bel}\mathbb{A}_3^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{6}\zeta + \frac{1}{2}\xi\alpha^2 - \frac{1}{2}\eta^2\xi + \frac{1}{2}\zeta^2\xi + \frac{1}{2}\zeta\xi^2 + \frac{1}{2}\zeta\xi \\ & + \frac{1}{2}\xi^2\alpha + \frac{1}{2}\xi\alpha\lambda + \alpha\zeta\xi + \frac{1}{6}\xi^3 + \frac{1}{2}\alpha\zeta^2 \\ & + \frac{1}{2}\alpha^2\zeta - \frac{1}{2}\zeta\eta^2 + \frac{1}{2}\zeta\alpha + \frac{1}{2}\alpha\zeta\lambda + \frac{1}{6}\alpha^3 \\ & + \frac{1}{6}\alpha\lambda^2 + \frac{1}{2}\alpha^2\lambda + \frac{1}{6}\lambda\alpha - \frac{1}{2}\eta^2\alpha + \frac{1}{6}\zeta^3 + \frac{1}{2}\zeta^2, \end{aligned}$$

$$\begin{aligned} {}_{Bel}\mathbb{A}_4^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{24}\zeta + \frac{1}{24}\xi^4 + \frac{1}{6}\zeta\xi + \frac{1}{2}\zeta^2\xi + \frac{1}{4}\zeta\xi^2 + \frac{1}{4}\xi^2\alpha^2 \\ & + \frac{1}{6}\zeta^3\xi + \frac{1}{4}\zeta^2\xi^2 + \frac{1}{6}\zeta\xi^3 + \frac{1}{6}\alpha^3\xi - \frac{1}{4}\eta^2\xi^2 \\ & + \frac{1}{6}\xi^3\alpha + \frac{1}{6}\alpha\lambda^2\xi + \frac{1}{2}\alpha^2\lambda\xi - \frac{1}{2}\zeta\eta^2\xi + \frac{1}{2}\alpha\zeta\xi^2 \\ & + \frac{1}{2}\alpha\zeta^2\xi + \frac{1}{2}\alpha^2\zeta\xi + \frac{1}{4}\xi^2\alpha\lambda - \frac{1}{2}\alpha\eta^2\xi + \frac{1}{6}\xi\alpha\lambda \\ & + \frac{1}{2}\alpha\zeta\xi + \frac{1}{4}\alpha\zeta^2\lambda + \frac{1}{2}\alpha^2\zeta\lambda + \frac{1}{6}\alpha\zeta\lambda^2 - \frac{1}{2}\alpha\zeta\eta^2 \\ & - \frac{1}{4}\alpha\lambda\eta^2 + \frac{1}{6}\alpha\zeta^3 + \frac{1}{4}\alpha^2\zeta^2 - \frac{1}{4}\zeta^2\eta^2 + \frac{1}{6}\alpha^3\zeta \\ & + \frac{1}{24}\alpha\lambda^3 + \frac{7}{24}\alpha^2\lambda^2 + \frac{1}{4}\alpha^3\lambda - \frac{1}{4}\alpha^2\eta^2 + \frac{1}{2}\alpha\zeta^2 \\ & + \frac{1}{4}\alpha^2\zeta - \frac{1}{4}\zeta\eta^2 + \frac{1}{6}\zeta\alpha + \frac{1}{2}\alpha\zeta\lambda + \frac{5}{12}\alpha\zeta\lambda \\ & + \frac{1}{24}\alpha^4 + \frac{1}{24}\zeta^4 + \frac{1}{6}\alpha\lambda^2 + \frac{1}{6}\alpha^2\lambda + \frac{1}{24}\lambda\alpha \\ & + \frac{1}{4}\zeta^3 + \frac{7}{24}\zeta^2 + \frac{1}{24}\eta^4, \end{aligned}$$

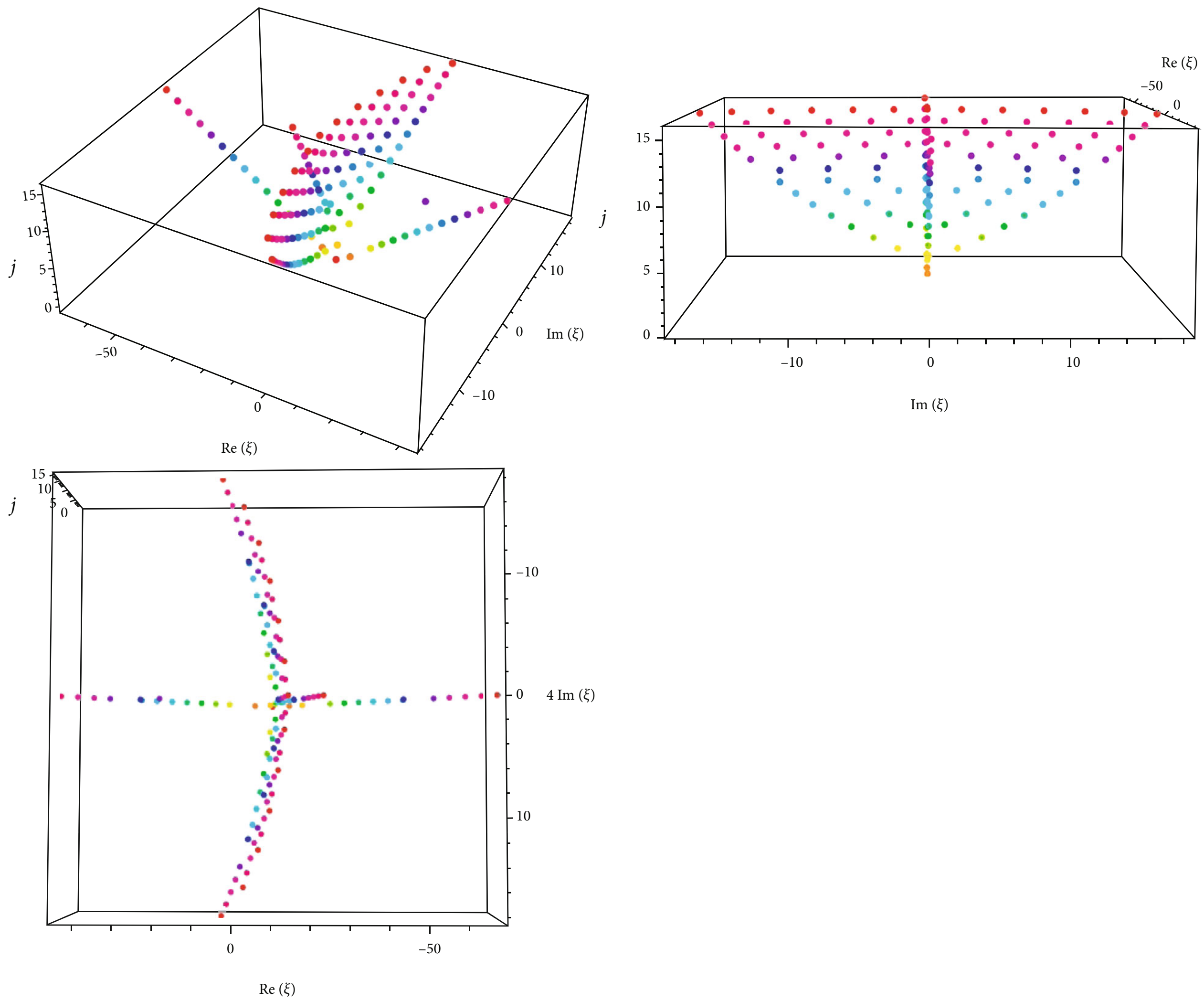
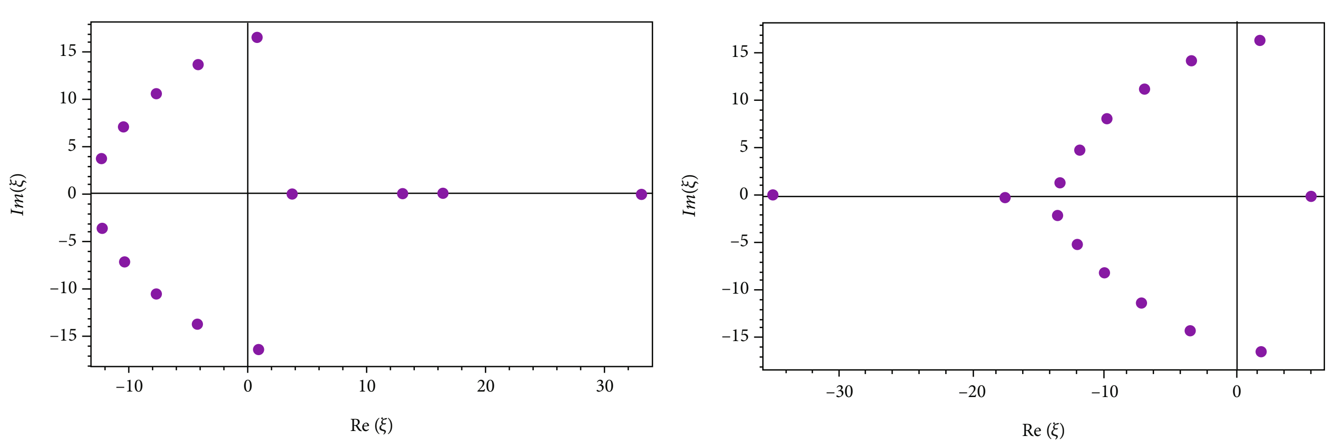


FIGURE 3: Stacking structure zeros of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5; 2)$, $j \leq 1 \leq 16$.



(a) Zeros of ${}_{Bel}\mathbb{A}_{15}^{(4,s)}(\xi, 6, 5; 2)$

(b) Zeros of ${}_{Bel}\mathbb{A}_{16}^{(4,s)}(\xi, 6, 5; 2)$

FIGURE 4: Graphic behavior of the zeros of ${}_{Bel}\mathbb{A}_j^{(4,s)}(\xi, 6, 5; 2)$ for $j = 15, 16$.

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_5^{(\alpha,c)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{120}\zeta + \frac{1}{120}\xi^5 + \frac{1}{24}\zeta\xi + \frac{7}{24}\zeta^2\xi + \frac{1}{12}\zeta\xi^2 \\
& + \frac{1}{4}\zeta^3\xi + \frac{1}{4}\zeta^2\xi^2 + \frac{1}{12}\zeta\xi^3 + \frac{1}{12}\xi^3\alpha^2 + \frac{1}{24}\eta^4\xi \\
& + \frac{1}{24}\zeta^4\xi + \frac{1}{12}\zeta^3\xi^2 + \frac{1}{12}\zeta^2\xi^3 + \frac{1}{24}\zeta\xi^4 + \frac{1}{12}\alpha^3\xi^2 \\
& + \frac{1}{24}\alpha^4\xi - \frac{1}{12}\eta^2\xi^3 + \frac{1}{24}\xi^4\alpha + \frac{1}{4}\alpha^2\zeta\xi^2 + \frac{1}{6}\alpha^3\zeta\xi \\
& - \frac{1}{4}\zeta^2\eta^2\xi + \frac{1}{4}\alpha\zeta^2\xi^2 + \frac{1}{6}\alpha\zeta^3\xi + \frac{1}{4}\alpha^2\zeta^2\xi + \frac{1}{12}\xi^3\alpha\lambda \\
& + \frac{7}{24}\alpha^2\lambda^2\xi + \frac{1}{4}\alpha^3\lambda\xi - \frac{1}{4}\alpha^2\eta^2\xi + \frac{1}{24}\alpha\lambda^3\xi + \frac{1}{6}\alpha\zeta\xi^3 \\
& - \frac{1}{4}\zeta\eta^2\xi^2 + \frac{1}{4}\alpha^2\lambda\xi^2 + \frac{1}{12}\alpha\lambda^2\xi^2 - \frac{1}{4}\alpha\eta^2\xi^2 + \frac{1}{6}\alpha\lambda^2\xi \\
& + \frac{1}{6}\alpha^2\lambda\xi - \frac{1}{4}\zeta\eta^2\xi + \frac{1}{4}\alpha\zeta\xi^2 + \frac{1}{2}\alpha\zeta^2\xi + \frac{1}{4}\alpha^2\zeta\xi \\
& + \frac{1}{12}\xi^2\alpha\lambda + \frac{1}{24}\xi\alpha\lambda + \frac{1}{6}\alpha\zeta\xi + \frac{1}{24}\alpha\zeta^4 + \frac{1}{12}\alpha^2\zeta^3 \\
& - \frac{1}{12}\zeta^3\eta^2 + \frac{1}{12}\alpha^3\zeta^2 + \frac{1}{24}\alpha^4\zeta + \frac{1}{24}\zeta\eta^4 + \frac{1}{120}\alpha\lambda^4 \\
& + \frac{1}{8}\alpha^2\lambda^3 + \frac{5}{24}\alpha^3\lambda^2 + \frac{1}{12}\alpha^4\lambda - \frac{1}{12}\alpha^3\eta^2 + \frac{1}{12}\alpha\zeta^3\lambda \\
& + \frac{1}{4}\alpha^2\zeta^2\lambda + \frac{1}{12}\alpha\zeta^2\lambda^2 - \frac{1}{4}\alpha\zeta^2\eta^2 + \frac{1}{4}\alpha^3\zeta\lambda \\
& + \frac{7}{24}\alpha^2\zeta\lambda^2 - \frac{1}{4}\alpha^2\zeta\eta^2 + \frac{1}{24}\alpha\zeta\lambda^3 - \frac{1}{4}\alpha^2\lambda\eta^2 \\
& - \frac{1}{12}\alpha\lambda^2\eta^2 + \frac{1}{3}\alpha\zeta^2\lambda + \frac{5}{12}\alpha^2\beta\lambda + \frac{1}{4}\alpha\zeta\lambda^2 - \frac{1}{4}\alpha\zeta\eta^2 \\
& - \frac{1}{12}\alpha\lambda\eta^2 + \frac{1}{4}\alpha\zeta^3 + \frac{1}{4}\alpha^2\zeta^2 - \frac{1}{4}\zeta^2\eta^2 \\
& + \frac{1}{12}\alpha^3\zeta + \frac{11}{120}\alpha\lambda^3 + \frac{1}{4}\alpha^2\lambda^2 + \frac{1}{12}\alpha^3\lambda \\
& + \frac{7}{24}\alpha\zeta^2 + \frac{1}{12}\alpha^2\zeta - \frac{1}{12}\zeta\eta^2 + \frac{1}{24}\alpha\zeta \\
& + \frac{1}{4}\alpha\zeta^2\lambda\xi + \frac{1}{2}\alpha^2\zeta\lambda\xi + \frac{1}{6}\alpha\zeta\lambda^2\xi + \frac{1}{4}\alpha\zeta\lambda\xi^2 \\
& - \frac{1}{2}\alpha\zeta\eta^2\xi - \frac{1}{4}\alpha\lambda\eta^2\xi + \frac{5}{12}\alpha\zeta\lambda\xi - \frac{1}{4}\alpha\zeta\lambda\eta^2 \\
& + \frac{5}{24}\alpha\zeta\lambda + \frac{1}{120}\alpha^5 + \frac{1}{120}\zeta^5 + \frac{1}{12}\zeta^4 + \frac{1}{24}\eta^4\alpha \\
& + \frac{11}{120}\alpha\lambda^2 + \frac{1}{24}\alpha^2\lambda + \frac{1}{120}\lambda\alpha + \frac{5}{24}\zeta^3 + \frac{1}{8}\zeta^2,
\end{aligned} \tag{63}
$$

while the first five parametric kinds of ${}_{Bel}\mathbb{A}_j^{(\alpha,s)}(\xi,\eta,\zeta|\lambda)$ are

$$
{}_{Bel}\mathbb{A}_0^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = 0,
$$

$$
{}_{Bel}\mathbb{A}_1^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \eta,
$$

$$
{}_{Bel}\mathbb{A}_2^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = (\xi+\zeta+\alpha)\eta,
$$

$$
{}_{Bel}\mathbb{A}_3^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = \frac{1}{6}\eta\left(3\alpha^2 + 6\zeta\alpha + 3\alpha\lambda + 6\xi\alpha + 3\zeta^2 + 6\zeta\xi - \eta^2 + 3\xi^2 + 3\zeta\right),
$$

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_4^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{6}\eta\Big(\alpha^3 + 3\alpha^2\zeta + 3\alpha^2\lambda + 3\xi\alpha^2 + 3\alpha\zeta^2 \\
& + 3\alpha\zeta\lambda + 6\alpha\zeta\xi + \alpha\lambda^2 + 3\xi\alpha\lambda - \eta^2\alpha + 3\xi^2\alpha \\
& + \zeta^3 + 3\zeta^2\xi - \zeta\eta^2 + 3\zeta\xi^2 - \eta^2\xi + \xi^3 + 3\zeta\alpha \\
& + \alpha\lambda + 3\zeta^2 + 3\zeta\xi + \zeta\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_5^{(\alpha,s)}(\xi,\eta,\zeta|\lambda) = {} & \frac{1}{120}\eta\big(5\zeta + 20\alpha\lambda^2\xi + 60\alpha^2\lambda\xi - 20\zeta\eta^2\xi \\
& + 60\alpha\zeta\xi^2 + 60\alpha\zeta^2\xi + 60\alpha^2\zeta\xi + 30\xi^2\alpha\lambda \\
& - 20\alpha\eta^2\xi + 5\xi^4 + 5\alpha^4 + 20\zeta\xi + 60\zeta^2\xi \\
& + 30\zeta\xi^2 + 30\xi^2\alpha^2 + 20\zeta^3\xi + 30\zeta^2\xi^2 \\
& + 20\zeta\xi^3 + 20\alpha^3\xi - 10\eta^2\xi^2 + 20\xi^3\alpha \\
& + 20\alpha\zeta^3 + 30\alpha^2\zeta^2 - 10\zeta^2\eta^2 + 20\alpha^3\zeta \\
& + 5\alpha\lambda^3 + 35\alpha^2\lambda^2 + 30\alpha^3\lambda - 10\alpha^2\eta^2 \\
& + 60\alpha\zeta^2 + 30\alpha^2\zeta - 10\zeta\eta^2 + 20\alpha\lambda^2 \\
& + 20\alpha^2\lambda + 20\zeta\alpha + 5\alpha\lambda + 20\xi\alpha\lambda + 60\alpha\zeta\xi \\
& + 30\alpha\zeta^2\lambda + 60\alpha^2\zeta\lambda + 20\alpha\zeta\lambda^2 - 20\alpha\zeta\eta^2 \\
& - 10\alpha\lambda\eta^2 + 50\alpha\zeta\lambda + 5\zeta^4 + 30\zeta^3 + 35\zeta^2 \\
& + \eta^4 + 60\alpha\zeta\lambda\xi\big).
\end{aligned} \tag{64}
$$

For $1 \le j \le 16$, the complex and real zero values of ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5\,; 2)$ are showed in Table 1.

Figure 1 shows the plots for some parametric cosine Frobenius-type Eulerian polynomials.

Figure 2 shows the structure of real zeros of the parametric cosine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,c)}(\xi, 6, 5\,; 2)$, with $j \le 1 \le 16$.

Figure 3 shows the stacking structure zeros of the parametric cosine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,c)}$ $(\xi, 6, 5\,; 2)$, with $j \le 1 \le 16$.

Finally, Figure 4 shows the graphic behavior of the zeros of the parametric sine Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_j^{(4,s)}(\xi, 6, 5\,; 2)$ for $j = 15, 16$.

4. Conclusion

Our paper introduced sine and cosine Bell-based Frobenius-type Eulerian polynomials and analyzed their properties by providing several relations and applications. Also, various formulas and properties including differentiation rules, addition formulas, relations, and summation formulas have been investigated. Moreover, after determining the first few zero values of the Eulerian polynomials, we have drawn graphical representations of these zero values.

It is possible that this paper's idea can be applied to polynomials that are similar and these polynomials have potential applications in other fields of science in addition to the applications at the end of the article. We will continue to explore this opinion in various directions in our next scientific works to advance the purpose of this article.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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