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Analytical solution proposal for fast numerical algorithm in special structured higher order differential equations

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Abstract

We suggest a practical method for obtaining the particular solution of non-homogeneous higher order linear differential equations with constant coefficients. The proposed method can be applied directly and simply to such problems. We revealed that is valid for the different type of problem by using sample solutions. This simple analytical solution that we have introduced will help to create a fast numerical algorithm for computers and thus simplify the numerical solutions of higher order physical problems.

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1. Introduction

Differential equations are vital argument for technical and fundamental sciences. Ordinary differential equations with constant coefficients provide practical utility owing to their mathematical controllable to explain natural phenomena in many areas of science and engineering [1,2]. For example, they are widely used to explain the event of electromagnetic, sound and water wave. Such waves are formulated by homogeneous or non-homogeneous differential equations with higher order linear constant coefficients depend upon source of the phenomena [3,4,5,6,7,8,9,10].

We focused on the solutions of the special equation type of non-homogeneous differential equations with high order linear constant coefficients. Solution of the differential equations with this special structure are very intractable to obtain its particular solution, since it requires to cope with several consecutive integral, multi-variable undefined linear equations system and integration of such unknown coefficients in the form of derivative functions [1,2,11,12].

In this work, without solving integral and multivariable linear equation system, a short formula and practices solution technique can be applied to handled problem directly is expressed. The considered differential equation is described by

$$(D^2+a^2)^n y = k_1 \sin ax + k_2 \cos ax, \quad (1)$$

where $n \in Z^+, a, k_1, k_2 \in R, D^2 = \frac{d^2}{dx^2}$. The general solution of a linear differential equation as follows:

$$y = y_h + y_p, \quad (2)$$

where y_h represents the solution of the homogeneous equation and y_p represents also a particular solution for the non-homogeneous differential equations. The solution of discussed problem of homogeneous equation is easy and obtained by the solution of the auxiliary equation. $(D^2+a^2)^n y_h = 0$ is obtained by the following:

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$$y_h = \sum_{j=0}^{n-1} x^j \left(C_j \cos ax + C_{j+n} \sin ax \right). \tag{3}$$

The solution to the homogeneous equation can be found as easily seen. Particular solutions of such equations are written in generally as follows:

$$y_p = \frac{1}{F(D)} Q(x) \tag{4}$$

where $F(D)$ is real coefficient polynomial of D. In order to acquire the particular solution of the proposed differential equation, we need the following four lemmas. Four lemmas have been proven below.

Lemma 1.1. If $F(D) = \sum_{j=0}^{n} a_j D^j, a_j \in R$ and $F(D)y = e^{iax}, i^2 = -1, F(ia) \neq 0,$ particular solution is

$$y_P = \frac{e^{iax}}{F(ia)}. \tag{5}$$

Proof.

Let $y = e^{iax}$. Then,

$Dy = iae^{iax}$

$D^2 y = (ia)^2 e^{iax}$

$D^3 y = (ia)^3 e^{iax}$

...

$D^n y = (ia)^n e^{iax}$.

Hence

$a_0 y = a_0 e^{iax}$

$a_1 Dy = a_1 (ia) e^{iax}$

$a_2 D^2 y = a_2 (ia)^2 e^{iax}$

$a_3 D^3 y = a_3 (ia)^3 e^{iax}$

...

$a_n D^n y = a_n (ia)^n e^{iax}$.

If these equations are added to the side-by-side,

$a_0 y + a_1 Dy + a_2 D^2 y + a_3 D^3 y + \ldots + a_n D^n y = a_0 e^{iax} + a_1 (ia) e^{iax} + a_2 (ia)^2 e^{iax} + a_3 (ia)^3 e^{iax} + \ldots + a_n (ia)^n e^{iax}$

we get

$$\left(\sum_{j=0}^{n} a_j D^j \right) y = \left(\sum_{j=0}^{n} a_j (ia)^j \right) e^{iax}.$$

If we denote

$$F(D)y = F(ia)e^{iax}$$

and

$$\frac{e^{iax}}{F(D)}=\frac{y}{F(ia)}$$

and also

$$y_p=\frac{e^{iax}}{F(D)}$$

we get the particular solution as below,

$$y_p=\frac{e^{iax}}{F(ia)}\,.$$

Lemma 1.2. If, $F(D)=\sum_{j=0}^{n}a_jD^j,a_j\in R$ and $F(D)y=e^{-iax},\ i^2=-1,\ F(-ia)\neq 0$, particular solution is

$$y_p=\frac{e^{-iax}}{F(-ia)} \quad (6)$$

Proof.

Let $y=e^{-iax}$. Therefore,

$Dy=-iae^{-iax}$

$D^2y=(-ia)^2e^{-iax}$

$D^3y=(-ia)^3e^{-iax}$

...

$D^ny=(-ia)^ne^{-iax}$.

Hence

$a_0y=a_0e^{-iax}$

$a_1Dy=a_1(-ia)e^{-iax}$

$a_2D^2y=a_2(-ia)^2e^{-iax}$

$a_3D^3y=a_3(-ia)^3e^{-iax}$

...

$a_nD^ny=a_n(-ia)^ne^{-iax}$.

Here, if these equations are adjoined to each other,

$a_0y+a_1Dy+a_2D^2y+\ldots+a_nD^ny=a_0e^{-iax}+a_1(-ia)e^{-iax}+a_2(-ia)^2e^{-iax}+\ldots+a_n(-ia)^ne^{-iax}$

We get

$$\left(\sum_{j=0}^{n}a_jD^j\right)y=\left(\sum_{j=0}^{n}a_j(-ia)^j\right)e^{-iax}$$

If we denote

$$F(D)y=F(-ia)e^{-iax}$$

and

$$\frac{e^{-iax}}{F(D)}=\frac{y}{F(-ia)}$$

and also

$$y_p=\frac{e^{-iax}}{F(D)}.$$

We get the particular solution as below,

$$y_p=\frac{e^{-iax}}{F(-ia)}.$$

Lemma 1.3. If $F_1(D)=\sum_{j=0}^{n}a_jD^j, a_j\in R$ and $F(D)=(D-ia)^r(D+ia)^rF_1(D)$, $F_1(ia)\neq 0$ and $F(D)y=e^{iax}$, $i^2=-1$, particular solution is

$$y_p=\frac{x^r}{r!}\frac{e^{iax}}{(2ia)^rF_1(ia)} \tag{7}$$

Proof.

Let $F(D)y=e^{iax}$.

Then, $(D-ia)^r(D+ia)^rF_1(D)y=e^{iax}$ and $(D-ia)^ry=\frac{e^{iax}}{(D+ia)^rF_1(D)}$. From Lemma 1.1.

$$(D-ia)^ry=\frac{e^{iax}}{(2ia)^rF_1(ia)}\ y_p=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}}{(D-ia)^r}\right)$$

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}\int e^{-iax}e^{iax}dx}{(D-ia)^{r-1}}\right)=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}x}{(D-ia)^{r-1}}\right)$$

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}\int e^{-iax}e^{iax}xdx}{(D-ia)^{r-2}}\right)=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}\frac{x^2}{2!}}{(D-ia)^{r-2}}\right)$$

...

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}\frac{x^{r-2}}{(r-2)!}}{(D-ia)^2}\right)$$

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}\int e^{-iax}e^{iax}\frac{x^{r-2}}{(r-2)!}dx}{(D-ia)}\right)=\frac{1}{(2ia)^rF_1(ia)}\left(\frac{e^{iax}\frac{x^{r-1}}{(r-1)!}}{(D-ia)}\right)$$

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(e^{iax}\int e^{-iax}e^{iax}\frac{x^{r-1}}{(r-1)!}dx\right)=\frac{1}{(2ia)^rF_1(ia)}\left(e^{iax}\int\frac{x^{r-1}}{(r-1)!}dx\right)$$

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(e^{iax}\int e^{-iax}e^{iax}\frac{x^{r-1}}{(r-1)!}dx\right),$$

we obtain the particular solution as below,

$$y_p=\frac{1}{(2ia)^rF_1(ia)}\left(e^{iax}\frac{x^r}{r!}\right) \tag{8}$$

Lemma 1.4. If $F(D)=\sum_{j=0}^{n} a_j D^j, a_j \in R$ and $F(D)=(D-ia)^r(D+ia)^r F_1(D)$, $F_1(-ia)\neq 0$ and $F(D)y=e^{-iax}$, $i^2=-1$, particular solution is

$$y_P=\frac{x^r}{r!}\frac{e^{-iax}}{(-2ia)^r F_1(-ia)} \tag{9}$$

Proof.

Let $F(D)y=e^{-iax}$ Therefore,

$(D-ia)^r(D+ia)^r F_1(D)y=e^{-iax}$

$(D+ia)^r y=\frac{e^{-iax}}{(D-ia)^r F_1(D)}$.

From Lemma 1.2.

$$(D+ia)^r y=\frac{e\text{-}iax}{(-2ia)^r F_1(-ia)}$$
$$y_p=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax}{(D+ia)^r}\right)$$
$$y_p=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax\, e^i ax\, e\text{-}iaxdx}{(D+ia)^r-1}\right)=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iaxx}{(D+ia)^r-1}\right)$$
$$y_p=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax\, e^i ax\, e\text{-}iaxxdx}{(D+ia)^r-2}\right)=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax\frac{x^2}{2!}}{(D+ia)^r-2}\right)$$
$$\cdots$$
$$y_p=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax\frac{x^r-2}{(r-2)!}}{(D+ia)^2}\right)$$
$$y_p=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax\, e^i ax\, e\text{-}iax\frac{x^r-2}{(r-2)!}dx}{(D+ia)}\right)=\frac{1}{(-2ia)^r F_1(-ia)}\left(\frac{e\text{-}iax\frac{x^r-1}{(r-1)!}}{(D+ia)}\right),$$

we obtain the particular solution as below,

$$y_p=\frac{1}{(-2ia)^r F_1(-ia)}\left(e^{-iax}\int e^{iax}e^{-iax}\frac{x^{r-1}}{(r-1)!}dx\right)=\frac{x^r}{r!}\frac{e^{-iax}}{(-2ia)^r F_1(-ia)}.$$

2. A Simple Approach to the Particular Solution, y_p

Particular solution of considered differential equations which is linear, constant coefficient, non-homogeneous and higher order differential equation is written as

$$y_P=\frac{1}{(D^2+a^2)^n}(k_1\sin(ax)+k_2\cos(ax)) \tag{10}$$

Using lemmas 1.1.,1.2.,1.3.,1.4. the Euler Identity by writing instead of $\sin ax$ $\cos ax$ in (10), it becomes the following:

$$
\begin{aligned}
y_p &= k_1\frac{\frac{e^{iax}-e^{-iax}}{2i}}{(D^2+a^2)^n}+k_2\frac{\frac{e^{iax}+e^{-iax}}{2}}{(D^2+a^2)^n}\\
&= k_1\frac{\frac{e^{iax}}{2i}}{(D^2+a^2)^n}+k_2\frac{\frac{e^{iax}}{2}}{(D^2+a^2)^n}+k_1\frac{\frac{-e^{-iax}}{2i}}{(D^2+a^2)^n}+k_2\frac{\frac{e^{-iax}}{2}}{(D^2+a^2)^n}\\
&= \left(\frac{k_1}{2i}+\frac{k_2}{2}\right)\left(\frac{e^{iax}}{(D^2+a^2)^n}\right)+\left(-\frac{k_1}{2i}+\frac{k_2}{}2\right)\left(\frac{e^{-iax}}{(D^2+a^2)^n}\right)\\
&= \left(\frac{k_1}{2i}+\frac{k_2}{2}\right)\frac{1}{(D-ia)^n}\left(\frac{e^{iax}}{(D+ia)^n}\right)+\left(-\frac{k_1}{2i}+\frac{k_2}{2}\right)\frac{1}{(D+ia)^n}\left(\frac{e^{-iax}}{(D-ia)^n}\right)\\
&= \left(\frac{k_1}{2i}+\frac{k_2}{2}\right)\frac{x^n}{n!}\left(\frac{e^{iax}}{(ia+ia)^n}\right)+\left(-\frac{k_1}{}2i+\frac{k_2}{2}\right)\frac{x^n}{n!}\left(\frac{e^{-iax}}{(-ia-ia)^n}\right)\\
&= \left(\frac{k_1}{2i}+\frac{k_2}{2}\right)\frac{x^n}{n!}\left(\frac{e^{iax}}{(2ia)^n}\right)+\left(-\frac{k_1}{}2i+\frac{k_2}{2}\right)\frac{x^n}{n!}\left(\frac{e^{-iax}}{(-2ia)^n}\right)\\
&= \frac{x^n}{2^na^nn!}\left(\left(\frac{k_1}{i}+k_2\right)\left(\frac{e^{iax}}{2(i)^n}\right)+\left(-\frac{k_1}{i}+k_2\right)\left(\frac{e^{-iax}}{2(-i)^n}\right)\right).
\end{aligned}
\quad (11)
$$

The particular solution is obtained in two ways depending on being even or odd of n. If n is even and $n=2m\,(m\in Z^{+})$

$$
\begin{aligned}
y_p &= \frac{x^{2m}}{2^{2m}a^{2m}2m!}\left(\left(\frac{k_1}{i}+k_2\right)\left(\frac{e^{iax}}{2(i)^{2m}}\right)+\left(-\frac{k_1}{i}+k_2\right)\left(\frac{e^{-iax}}{2(-i)^{2m}}\right)\right)\\
&= \frac{x^{2m}}{(-1)^m2^{2m}a^{2m}2m!}\left(\left(\frac{k_1}{i}+k_2\right)\left(\frac{e^{iax}}{2}\right)+\left(-\frac{k_1}{i}+k_2\right)\left(\frac{e^{-iax}}{2}\right)\right)\\
&= \frac{x^{2m}}{(-1)^m2^{2m}a^{2m}2m!}\left(k_1\frac{e^{iax}-e^{-iax}}{2i}+k_2\frac{e^{iax}+e^{-iax}}{2}\right).
\end{aligned}
\quad (12)
$$

Therefore, particular solution (10) for even values of n is obtained below:

$$
y_p=\frac{x^{2m}}{(-1)^m2^{2m}a^{2m}2m!}(k_1\text{sin}ax+k_2\text{cos}ax).
\quad (13)
$$

Using these result, we obtain the following general solution of (1) for even values of n as follows:

$$
y=(-1)^{\frac{n}{2}}\frac{x^n}{2^na^nn!}(k_1\text{sin}ax+k_2\text{cos}ax)+\sum_{j=0}^{n-1}x^j(C_j\text{cos}ax+C_{j+n}\text{sin}ax).
\quad (14)
$$

If n is odd and $n=2m+1\,(m\in Z^{+})$, hence, particular solution (10) is

$$
\begin{aligned}
y_p &= \frac{x^{2m+1}}{2^{2m+1}a^{2m+1}(2m+1)!}\left(\left(\frac{k_1}{i}+k_2\right)\left(\frac{e^{iax}}{2(i)^{2m+1}}\right)+\left(-\frac{k_1}{i}+k_2\right)\left(\frac{e^{-iax}}{2(-i)^{2m+1}}\right)\right)\\
&= \frac{x^{2m+1}}{2^{2m+1}a^{2m+1}(2m+1)!}\left(\frac{k_1}{i}\frac{e^{iax}}{2(i)^{2m+1}}+k_2\frac{e^{iax}}{2(i)^{2m+1}}-\frac{k_1}{i}\frac{e^{-iax}}{2(-i)^{2m+1}}+k_2\frac{e^{-iax}}{2(-i)^{2m+1}}\right)\\
&= \frac{x^{2m+1}}{2^{2m+1}a^{2m+1}(2m+1)!}\left(\frac{k_1}{i^2}\frac{e^{iax}}{2(i)^{2m}}+k_2\frac{e^{iax}}{2(i)^{2m}2i}+\frac{k_1}{i^2}\frac{e^{-iax}}{2(i)^{2m}2}-k_2\frac{e^{-iax}}{(i)^{2m}2i}\right)\\
&= \frac{x^{2m+1}}{2^{2m+1}a^{2m+1}(2m+1)!(i)^{2m}}\left(-k_1\frac{e^{iax}}{2}+k_2\frac{e^{iax}}{2i}-k_1\frac{e^{-iax}}{2}-k_2\frac{e^{-iax}}{2i}\right)\\
&= (-1)^m\frac{x^{2m+1}}{2^{2m+1}a^{2m+1}(2m+1)!}\left(-k_1\frac{e^{iax}+e^{-iax}}{2}+k_2\frac{e^{iax}-e^{-iax}}{}2i\right).
\end{aligned}
\quad (15)
$$

Therefore, particular solution (10) for odd values of n is obtained below:

$$
y_p=(-1)^m\frac{x^{2m+1}}{2^{2m+1}a^{2m+1}(2m+1)!}(-k_1\text{cos}ax+k_2\text{sin}ax).
\quad (16)
$$

Using these result, we obtain the following general solution of (1) for odd values of n as follows:

$$y=(-1)^{\frac{n-1}{2}}\frac{x^n}{2^n a^n (n)!}(-k_1\cos ax+k_2\sin ax)+\sum_{j=0}^{n-1}x^j(C_j\cos ax+C_{j+n}\sin ax). \quad (17)$$

3. Examples

Example 1

Consider such a differential equation $(D^2+1)^5 y=2\sin x-3\cos x$. Solution of this equation: firstly, we solve the homogen equation,

$$(D^2+1)^5 y=0(m^2+1)^5=0m_{1,2,3,4,5}=iy_1=\cos x, y_2=x\cos x, y_3=x^2\cos x, y_4=x^3\cos x, y_5=x^4\cos x$$
$$m_{6,7,8,9,10}=-iy_6=\cos x, y_7=x\cos x, y_8=x^2\cos x, y_9\%=x^3\cos x, y_{10}=x^4\cos x$$
$$y=y_h y\sum_{i=1}^{10}C_i y_i$$

Now, let's try to solve the non-homogeneous equation using the classical methods and then the formula we recommend.

(1) By using the method of consecutive integration

$$y_P=e^{ix}\int\int\int\int\int e^{-2ix}\int\int\int\int\int e^{ix}(2\sin x-3\cos x)(dx)^{10}$$

As can be seen, the solution of y_P with this method will be found by integrating ten times in succession. It is clear that integrating in this way is not quick and easy.

(2) By using the method of variation of parameters

Let us $y_P=\sum_{i=1}^{10}L_i y_i$. L_i functions are defined as:

$$\sum_{i=1}^{10}L_i' y_i=0,\ \sum_{i=1}^{10}L_i' y_i^{(1)}=0,\ \ldots,\ \sum_{i=1}^{10}L_i' y_i^{(8)}=0,\ \sum_{i=1}^{10}L_i' y_i^{(9)}=2\sin x-3\cos x$$

Solving above linear equation system, with the derivative functions of $L_i', (1\le i\le 10)$ using this derivative functions and following integration L_i yields

$$L_i=\int L_i' dx, (1\le i\le 10).$$

It can not be said that these two processes will be quick and easy.

(3) By using the method of undetermined coefficients

Let us $y_P=\left(\sum_{i=0}^{9}A_i x^i\right)\cos x+\left(\sum_{i=0}^{9}B_i x^i\right)\sin x$. In this equation, replaces $y, D_{x^2}y, \ldots, D_{x^{10}}y$ and its derivatives. With the identification of the two sides, $A_i, B_i (1\le i\le 10)$ coefficients are determined. It can also not be said that these two processes will be quick and easy.

(4) Finally, finding the particular solution by using recommended method in eq.17

Since $n=5$, the method recommended for the odd value in equation (17) is used. Therefore,

$$y_P=\frac{x^n}{(-1)^{\frac{n-1}{2}}2^n a^n n!}(-k_1\cos ax+k_2\sin ax)$$

For $a=1, k_1=2, k_2=-3, n=5$ yields

$$y_P = \frac{1}{3840} x^5 (- 2\cos x - 3\sin x).$$

Proof:

$$y = y_P$$

$$\begin{aligned}(D^2+1)^5 y &= (D^{10} + 5D^8 + 10D^6 + 10D^4 + 5D^2 + 1) y \\ &= D_{x^{10}} y + 5D_{x^8} y + 10D_{x^6} y + 10D_{x^4} y + 5D_{x^2} y + y\end{aligned}$$

$$y = \frac{1}{3840} x^5 (- 2\cos x - 3\sin x)$$

$$D_{x^2} y = \frac{1}{3840} x^3 (20x\sin x - 60\sin x - 30x\cos x - 40\cos x + 2x^2\cos x + 3x^2\sin x)$$

$$D_{x^4} y = \left(- \frac{1}{3840} \right) x \begin{pmatrix} 240\cos x + 360\sin x + 720x\cos x - 480x\sin x - 240x^2\cos x \\ - 60x^3\cos x + 2x^4\cos x - 360x^2\sin x + 40x^3\sin x + 3x^4\sin x \end{pmatrix}$$

$$D_{x^6} y = \frac{1}{3840} \begin{pmatrix} 1440\sin x - 2160\cos x + 3600x\cos x + 5400x\sin x + 3600x^2\cos x \\ - 600x^3\cos x - 90x^4\cos x + 2x^5\cos x - 2400x^2\sin x \\ - 900x^3\sin x + 60x^4\sin x + 3x^5\sin x \end{pmatrix}$$

$$D_{x^8} y = \left(- \frac{1}{3840} \right) \begin{pmatrix} 13\,440\sin x - 20\,160\cos x + 16\,800x\cos x + 25\,200x\sin x \\ + 10\,080x^2\cos x - 1120x^3\cos x - 120x^4\cos x + 2x^5\cos x \\ - 6720x^2\sin x - 1680x^3\sin x + 80x^4\sin x + 3x^5\sin x \end{pmatrix}$$

$$D_{x^{10}} y = \frac{1}{3840} \begin{pmatrix} 60\,480\sin x - 90\,720\cos x + 50\,400x\cos x + 75\,600x\sin x \\ + 21\,600x^2\cos x - 1800x^3\cos x - 150x^4\cos x + 2x^5\cos x \\ - 14\,400x^2\sin x - 2700x^3\sin x + 100x^4\sin x + 3x^5\sin x \end{pmatrix}$$

$$D_{x^{10}} y + 5D_{x^8} y + 10D_{x^6} y + 10D_{x^4} y + 5D_{x^2} y + y = 2\sin x - 3\cos x$$

Example 2

Consider such a differential equation

$$(D^2+1)^{10} y = 2\sin x - 3\cos x.$$

Since $n = 10$, the method recommended for the even value in equation (14) is used. Therefore,

$$y_P = \frac{x^n}{(- 1)^{\frac{n}{2}} 2^n a^n n!} (k_1 \sin ax + k_2 \cos ax)$$

For $a = 1, k_1 = 2, k_2 = - 3, n = 10$ gives

$$y_P = \left(- \frac{1}{3715\,891\,200} \right) x^{10} (2\sin x - 3\cos x)$$

Proof:

$$\begin{aligned}(D^2+1)^{10} y &= \begin{pmatrix} D^{20} + 10D^{18} + 45D^{16} + 120D^{14} + 210D^{12} + 252D^{10} \\ + 210D^8 + 120D^6 + 45D^4 + 10D^2 + 1 \end{pmatrix} y \\ &= D_{x^{20}} y + 10D_{x^{18}} y + 45D_{x^{16}} y + 120D_{x^{14}} y + 210D_{x^{12}} y + 252D_{x^{10}} y \\ &\quad + 210D_{x^8} y + 120D_{x^6} y + 45D_{x^4} y + 10D_{x^2} y + y\end{aligned}$$

$$y = - \frac{1}{2^{18}3^4 5^2 7} x^{10} (2\sin x - 3\cos x)$$

$$D_{x^2} y = - \frac{1}{2^{18}3^4 5^2 7} x^8 \begin{pmatrix} 180\sin x - 270\cos x + 40x\cos x \\ + 60x\sin x + 3x^2\cos x - 2x^2\sin x \end{pmatrix}$$

$$D_{x^4}y = -\frac{1}{2^{18}3^4 5^2 7}x^6\begin{pmatrix} 10\,080\sin x - 15\,120\cos x + 5760x\cos x \\ + 8640x\sin x + 1620x^2\cos x - 80x^3\cos x \\ - 3x^4\cos x - 1080x^2\sin x - 120x^3\sin x + 2x^4\sin x \end{pmatrix}$$

$$D_{x^6}y = \frac{1}{2^{18}3^4 5^2 7}x^4\begin{pmatrix} 453\,600\cos x - 302\,400\sin x - 362\,880x\cos x \\ - 544\,320x\sin x - 226\,800x^2\cos x + 28\,800x^3\cos x + 4050x^4\cos x \\ - 120x^5\cos x - 3x^6\cos x + 151\,200x^2\sin x + 43\,200x^3\sin x \\ - 2700x^4\sin x - 180x^5\sin x + 2x^6\sin x \end{pmatrix}$$

$$D_{x^8}y = \left(-\frac{1}{2^{18}3^4 5^2 7}\right)x^2\begin{pmatrix} 3628\,800\sin x - 5443\,200\cos x + 9676\,800x\cos x \\ + 14\,515\,200x\sin x + 12\,700\,800x^2\cos x - 3386\,880x^3\cos x \\ - 1058\,400x^4\cos x + 80\,640x^5\cos x + 7560x^6\cos x \\ - 8467\,200x^2\sin x - 160x^7\cos x - 5080\,320x^3\sin x - 3x^8\cos x \\ + 705\,600x^4\sin x + 120\,960x^5\sin x - 5040x^6\sin x \\ - 240x^7\sin x + 2x^8\sin x \end{pmatrix}$$

$$D_{x^{10}}y = \frac{1}{2^{18}3^4 5^2 7}\begin{pmatrix} 10\,886\,400\cos x - 7257\,600\sin x - 72\,576\,000x\cos x \\ - 108\,864\,000x\sin x - 244\,944\,000x^2\cos x + 145\,152\,000x^3\cos x \\ + 95\,256\,000x^4\cos x - 15\,240\,960x^5\cos x - 3175\,200x^6\cos x \\ + 163\,296\,000x^2\sin x + 172\,800x^7\cos x + 217\,728\,000x^3\sin x \\ + 12\,150x^8\cos x - 63\,504\,000x^4\sin x - 200x^9\cos x \\ - 22\,861\,440x^5\sin x - 3x^{10}\cos x + 2116\,800x^6\sin x \\ + 259\,200x^7\sin x - 8100x^8\sin x - 300x^9\sin x + 2x^{10}\sin x \end{pmatrix}$$

$$D_{x^{12}}y = -\frac{1}{2^{18}3^4 5^2 7}\begin{pmatrix} 718\,502\,400\cos x - 479\,001\,600\sin x - 1596\,672\,000x\cos x \\ - 2395\,008\,000x\sin x - 2694\,384\,000x^2\cos x + 958\,003\,200x^3\cos x \\ + 419\,126\,400x^4\cos x - 47\,900\,160x^5\cos x - 7484\,400x^6\cos x \\ + 1796\,256\,000x^2\sin x + 316\,800x^7\cos x + 1437\,004\,800x^3\sin x \\ + 17\,820x^8\cos x - 279\,417\,600x^4\sin x - 240x^9\cos x \\ - 71\,850\,240x^5\sin x - 3x^{10}\cos x + 4989\,600x^6\sin x \\ + 475\,200x^7\sin x - 11\,880x^8\sin x - 360x^9\sin x + 2x^{10}\sin x \end{pmatrix}$$

$$D_{x^{14}}y = \frac{1}{2^{18}3^4 5^2 7}\begin{pmatrix} 10\,897\,286\,400\cos x - 7264\,857\,600\sin x - 14\,529\,715\,200x\cos x \\ - 21\,794\,572\,800x\sin x - 16\,345\,929\,600x^2\cos x + 4151\,347\,200x^3\cos x \\ + 1362\,160\,800x^4\cos x - 121\,080\,960x^5\cos x - 15\,135\,120x^6\cos x \\ + 10\,897\,286\,400x^2\sin x + 524\,160x^7\cos x + 6227\,020\,800x^3\sin x \\ + 24\,570x^8\cos x - 908\,107\,200x^4\sin x - 280x^9\cos x \\ - 181\,621\,440x^5\sin x - 3x^{10}\cos x + 10\,090\,080x^6\sin x \\ + 786\,240x^7\sin x - 16\,380x^8\sin x - 420x^9\sin x + 2x^{10}\sin x \end{pmatrix}$$

$$D_{x^{16}}y = \left(-\frac{1}{3715\,891\,200}\right)\begin{pmatrix} 87\,178\,291\,200\cos x - 58\,118\,860\,800\sin x - 83\,026\,944\,000x\cos x \\ - 124\,540\,416\,000x\sin x - 70\,053\,984\,000x^2\cos x + 13\,837\,824\,000x^3\cos x \\ + 3632\,428\,800x^4\cos x - 264\,176\,640x^5\cos x - 27\,518\,400x^6\cos x \\ + 46\,702\,656\,000x^2\sin x + 806\,400x^7\cos x + 20\,756\,736\,000x^3\sin x \\ + 32\,400x^8\cos x - 2421\,619\,200x^4\sin x - 320x^9\cos x \\ - 396\,264\,960x^5\sin x - 3x^{10}\cos x + 18\,345\,600x^6\sin x \\ + 1209\,600x^7\sin x - 21\,600x^8\sin x - 480x^9\sin x + 2x^{10}\sin x \end{pmatrix}$$

$$D_{x^{18}}y=\frac{1}{3715\,891\,200}\left(\begin{array}{c}476\,367\,091\,200\cos x-317\,578\,060\,800\sin x-352\,864\,512\,000x\cos x\\ -529\,296\,768\,000x\sin x-238\,183\,545\,600x^2\cos x+38\,494\,310\,400x^3\cos x\\ +8420\,630\,400x^4\cos x-518\,192\,640x^5\cos x-46\,267\,200x^6\cos x\\ +158\,789\,030\,400x^2\sin x+1175\,040x^7\cos x+57\,741\,465\,600x^3\sin x\\ +41\,310x^8\cos x-5613\,753\,600x^4\sin x-360x^9\cos x\\ -777\,288\,960x^5\sin x-3x^{10}\cos x+30\,844\,800x^6\sin x\\ +1762\,560x^7\sin x-27\,540x^8\sin x-540x^9\sin x+2x^{10}\sin x\end{array}\right)$$

$$D_{x^{20}}y=-\frac{1}{2^{18}3^45^27}\left(\begin{array}{c}2011\,327\,718\,400\cos x-1340\,885\,145\,600\sin x-1218\,986\,496\,000x\cos x\\ -1828\,479\,744\,000x\sin x-685\,679\,904\,000x^2\cos x+93\,768\,192\,000x^3\cos x\\ +17\,581\,536\,000x^4\cos x-937\,681\,920x^5\cos x-73\,256\,400x^6\cos x\\ +457\,119\,936\,000x^2\sin x+1641\,600x^7\cos x+140\,652\,288\,000x^3\sin x\\ +51\,300x^8\cos x-11\,721\,024\,000x^4\sin x-400x^9\cos x\\ -1406\,522\,880x^5\sin x-3x^{10}\cos x+48\,837\,600x^6\sin x\\ +2462\,400x^7\sin x-34\,200x^8\sin x-600x^9\sin x+2x^{10}\sin x\end{array}\right)$$

$$D_{x^{20}}y+10D_{x^{18}}y+45D_{x^{16}}y+120D_{x^{14}}y+210D_{x^{12}}y+252D_{x^{10}}y+210D_{x^8}y120D_{x^6}y+45D_{x^4}y+10D_{x^2}y+y=2\sin x-3\cos x\,.$$

4. Concluding remarks

When using the classical analytical methods known in the literature for solving the high order linear differential equations, the resulting systems of integral and multivariable linear equations make the numerical calculations of the related physical problems difficult and complex algorithms. However, the method we propose above and proved its validity will remove these difficulties altogether. This simple analytical solution we have presented will help to create a rapid numerical algorithm for computers and thus simplify the numerical solution of high order physical problems.

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